Identifying transport structure: set-oriented FTLE, bifurcations of transfer operator modes, and predicting critical transitions

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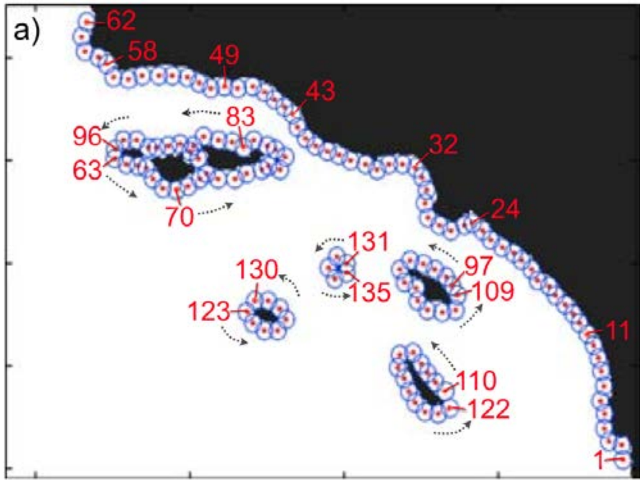
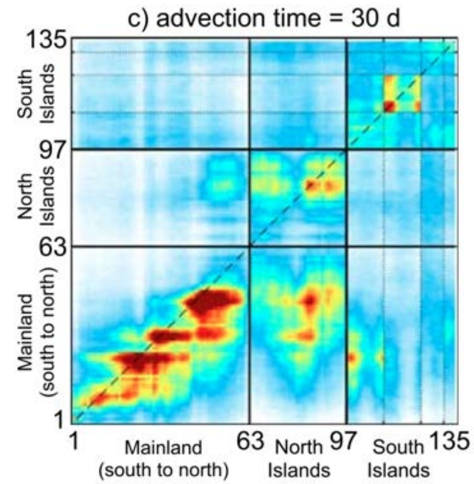
Join work with M. Stremler, D. Schmale, P. Vlachos, F. Lekien, A. BozorgMagham, S. Naik, P. Tallapragada, S. Raben, P. Grover, P. Kumar

SON 2013, TU Dresden (2 Oct 2013)

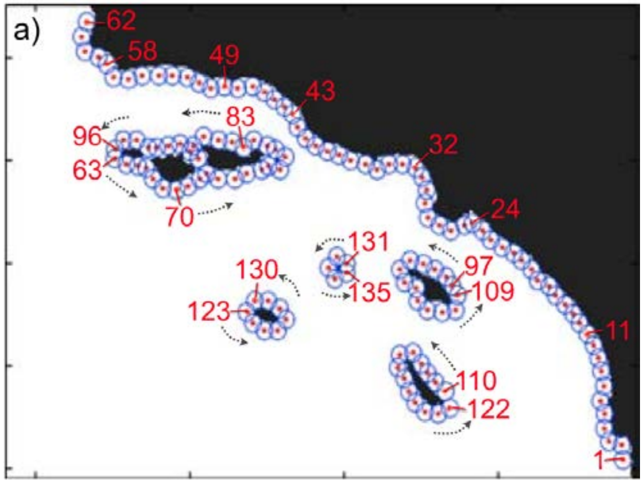
Southern California coast: highly mixed marine ecosystem

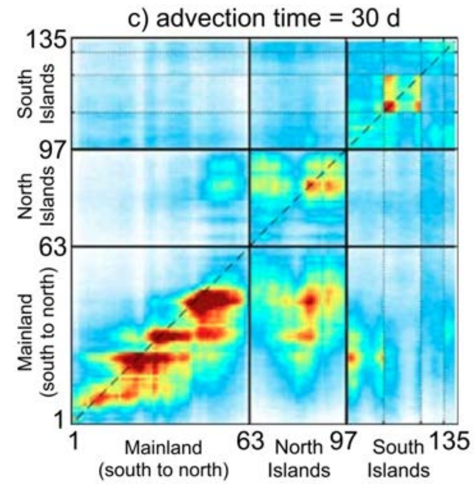




Fish larva transport, Cheryl Harrison, OSU; Harrison, Siegel, Mitarai [2013], Mitarai et al [2009]

Southern California coast: highly mixed marine ecosystem

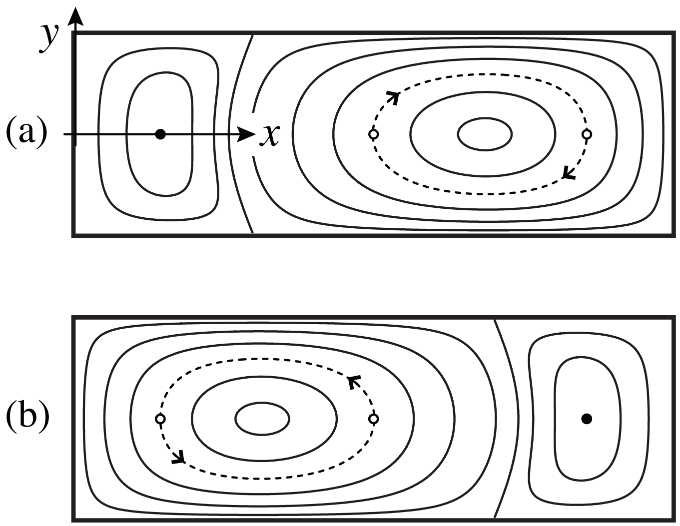




Sea surface height (streamlines if ocean surface velocity)

Ghost rods in microfluidic mixer

- Viscous flow in a 2D box (described by Mark Stremler on Monday)



streamlines for $\tau_f = 1$

tracer blob ($\tau_f > 1$)

- piecewise constant vector field (piecewise steady flow)

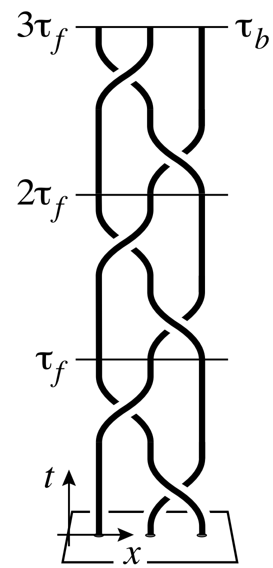
  $t \in [n\tau_f, (n+1)\tau_f/2)$, top streamline pattern

  $t \in [(n+1)\tau_f/2, (n+1)\tau_f)$, bottom streamline pattern

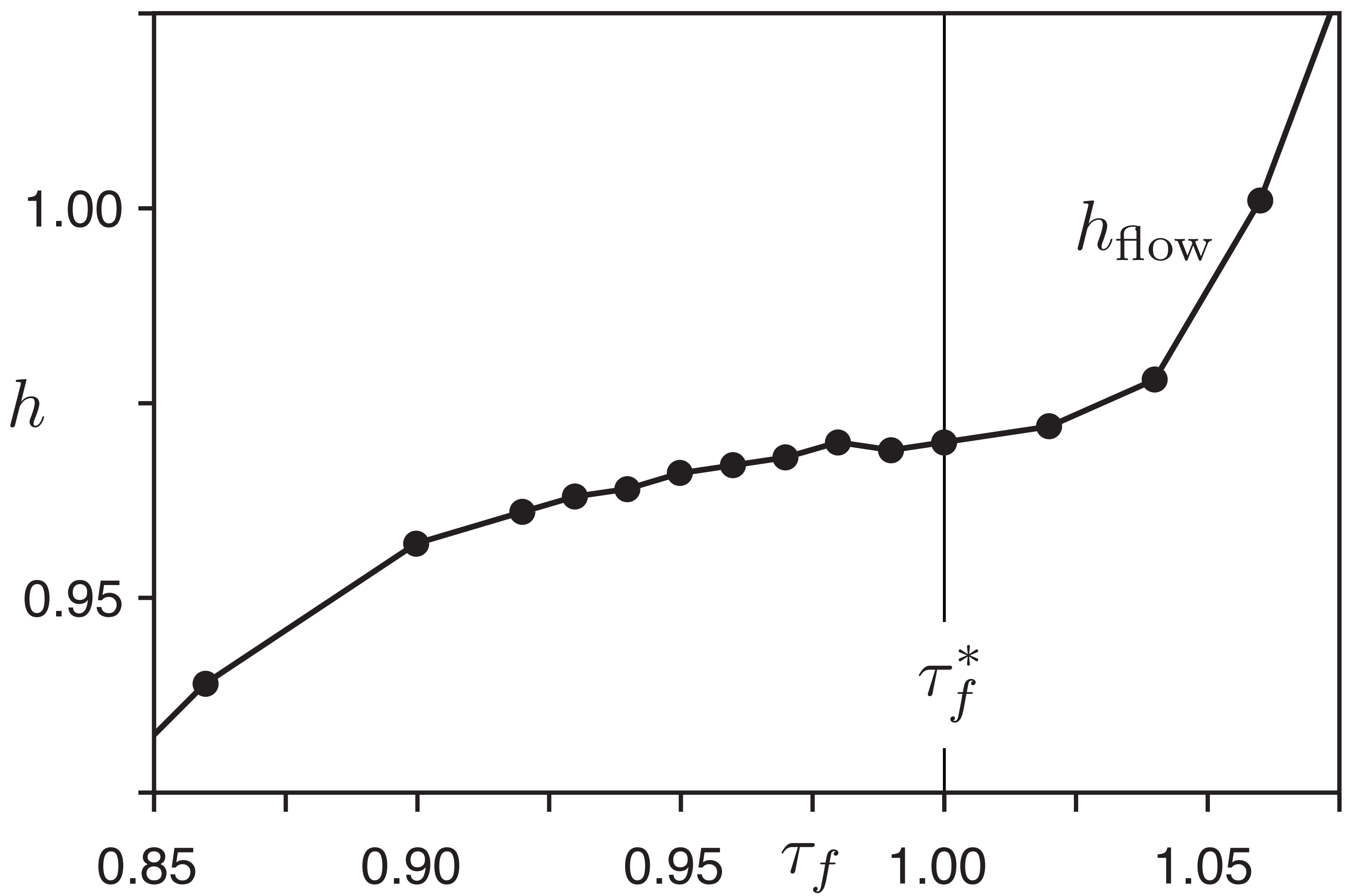
- System has parameter τ_f, which we treat as a bifurcation parameter — critical point $\tau_f^* = 1$

Ghost rods in microfluidic mixer

- For $\tau_f = 1$, braid on 3 strands act as 'ghost rods' stirring the fluid
- Their braid has $h_{\mathrm{TN}} = 0.962$ from Thurston-Nielsen Classification Theorem
- Actual for flow $h_{\mathrm{flow}} = 0.964$
- $\Rightarrow$ h_{TN} is an excellent lower bound

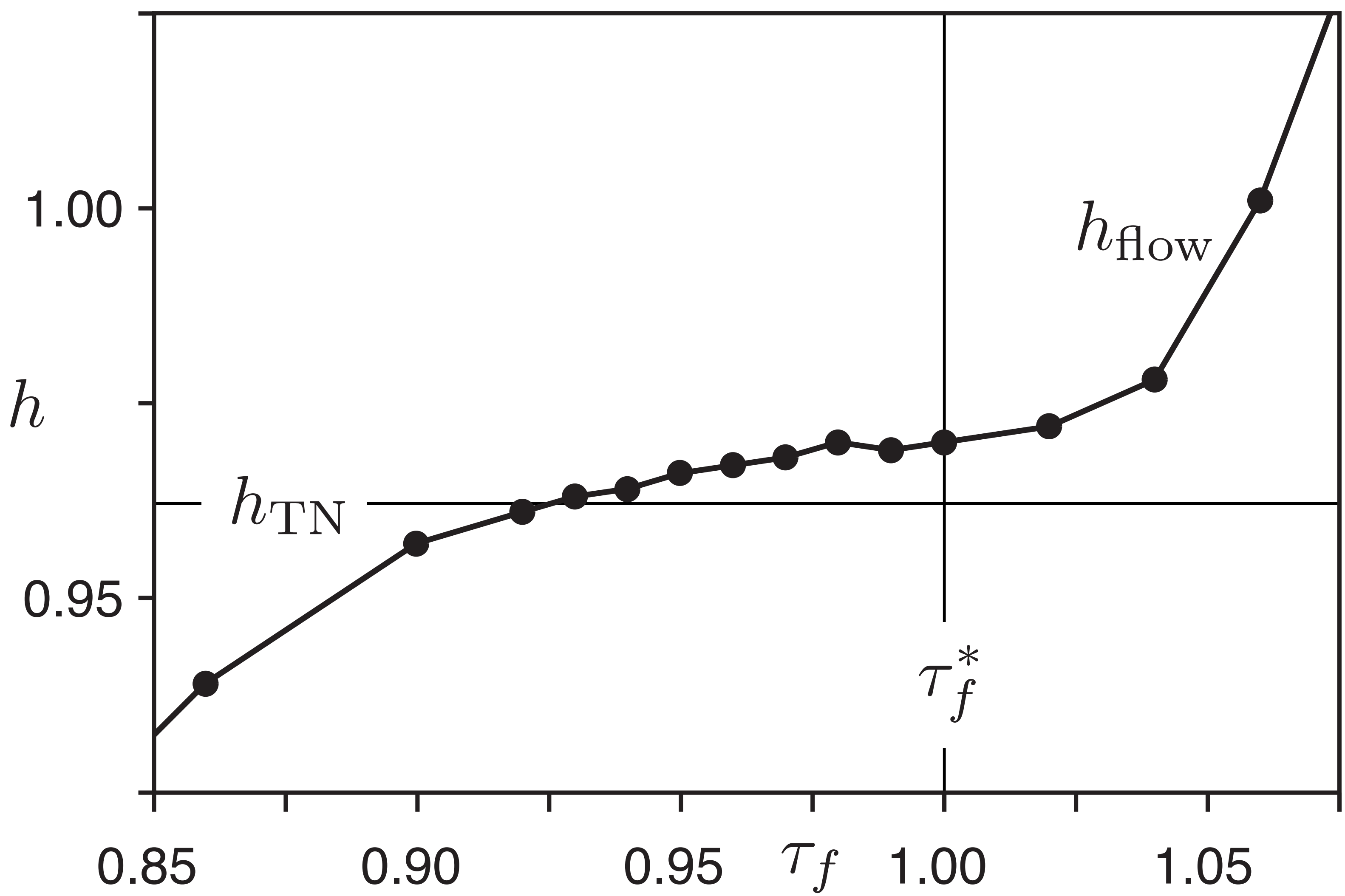


Topological entropy continuity across critical point



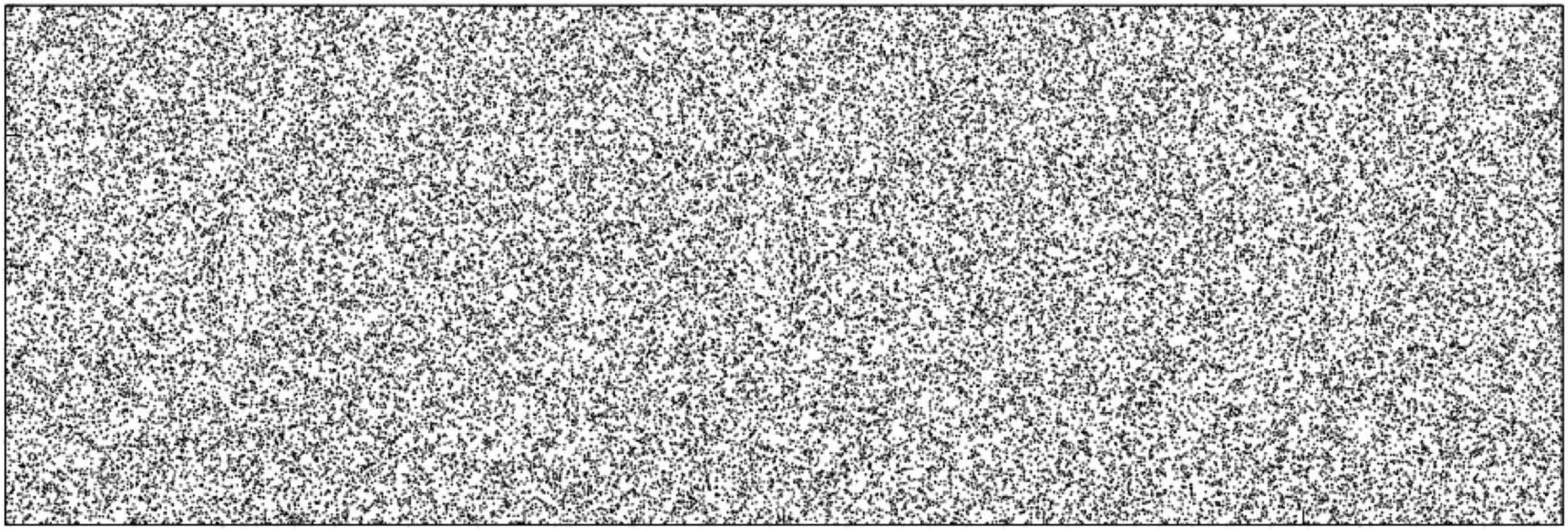
topological entropy as a function of τ_f

Topological entropy continuity across critical point



topological entropy as a function of τ_f

Identifying 'ghost rods'?



Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

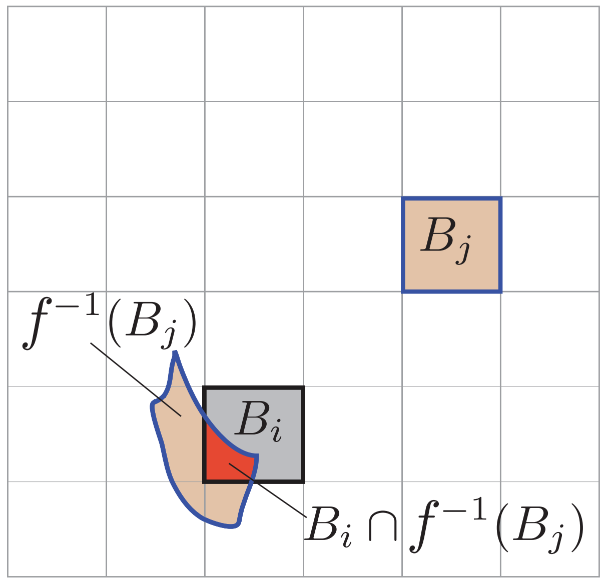
- Note the absence of any elliptical islands
- No periodic orbits of low period
- Is the phase space featureless?

Almost-invariant / almost-cyclic set approach

- Identify **almost-invariant sets** (AISs) using probabilistic point of view
- Relatedly, **almost-cyclic sets** (ACSs)[1]
- Create box partition of phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, with q large
- Consider a q-by-q **transition (Ulam) matrix**, P, where

$$P_{ij} = \frac{m(B_i \cap f^{-1}(B_j))}{m(B_i)},$$

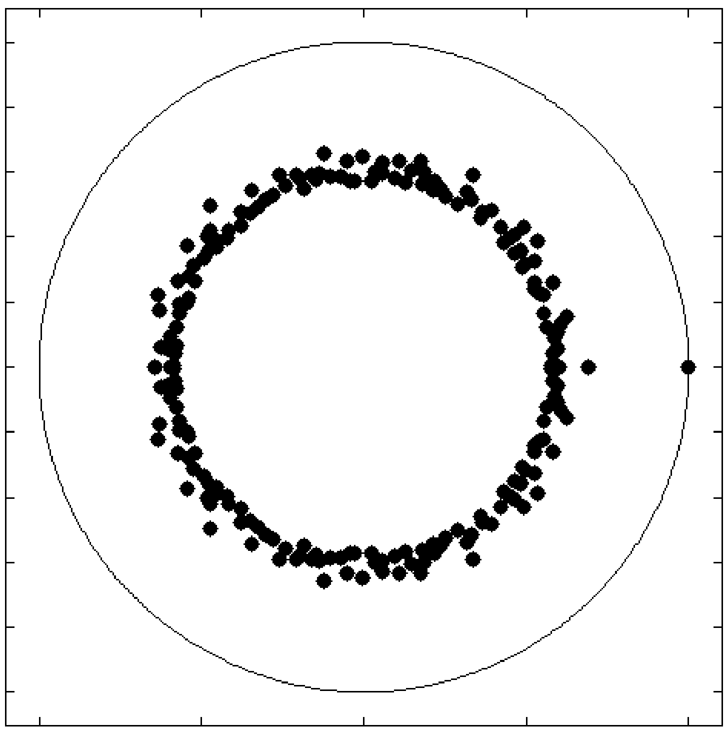
the *transition probability* from B_i to B_j using, e.g., $f = \phi_t^{t+T}$, computed numerically



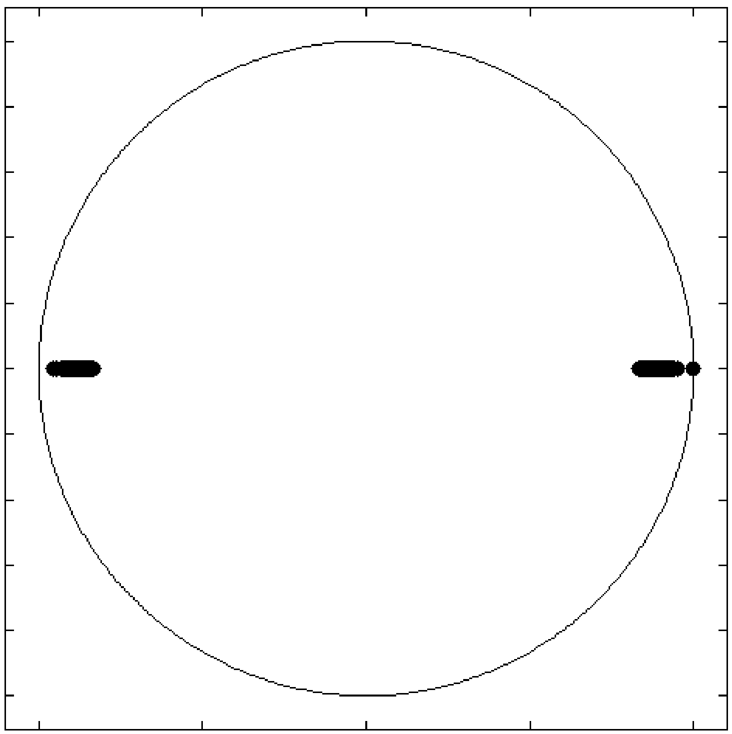
- P approximates $\mathcal{P}$, Perron-Frobenius operator — which evolves densities, ν, over one iterate of f, as $\mathcal{P}\nu$
- Typically, we use a reversibilized operator R, obtained from P

[1] Dellnitz & Junge [1999], Froyland & Dellnitz [2003]

Identifying AISs by spectrum-partitioning



top 200 eigenvalues of P

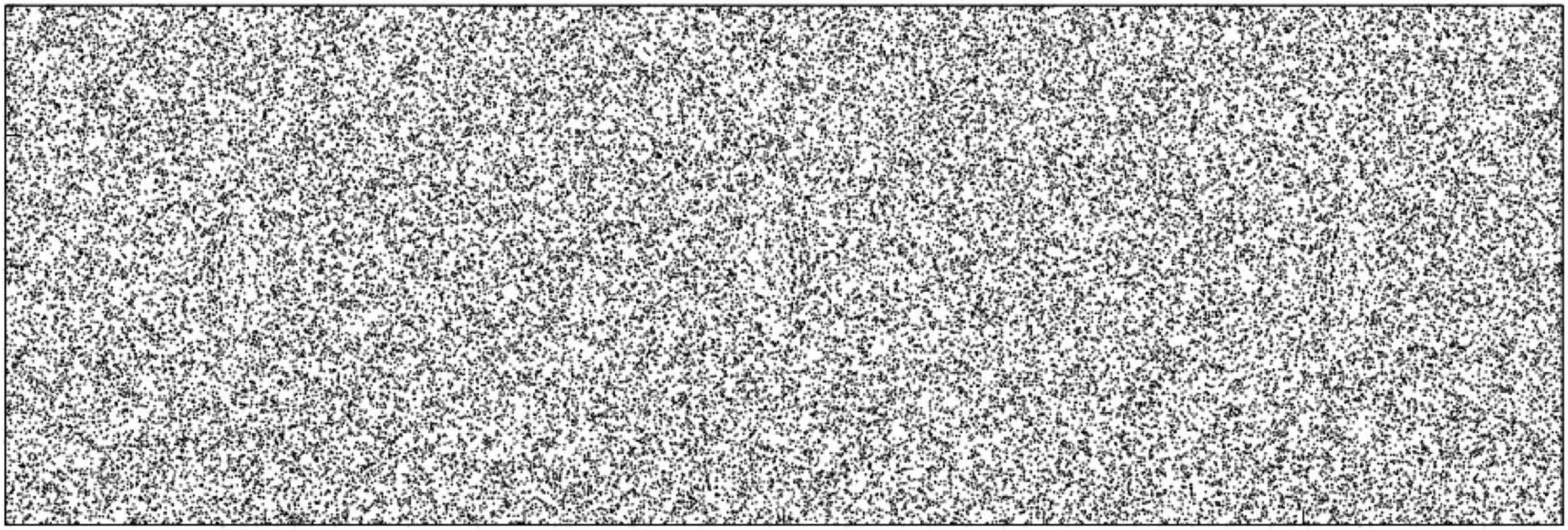


top 200 eigenvalues of R

- **Invariant** densities are those fixed under P, $P\nu = \nu$, i.e., eigenvalue 1
- The other real eigenvalues can identify **almost-invariant** sets

Dellnitz, Froyland, Sertl [2000] Nonlinearity

Identifying 'ghost rods': almost-cyclic sets

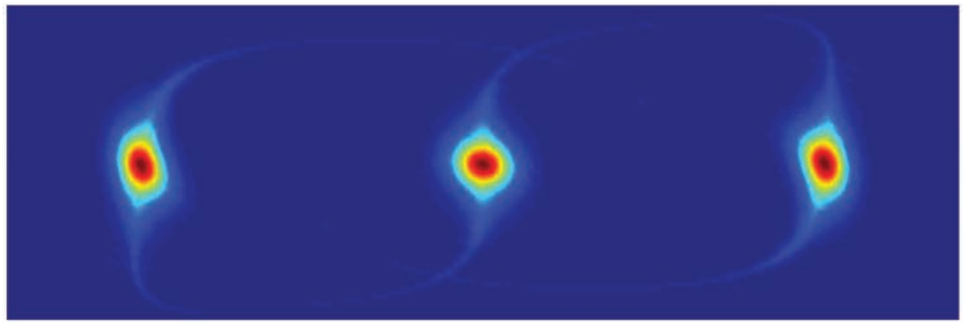


Poincaré section with no obvious structure

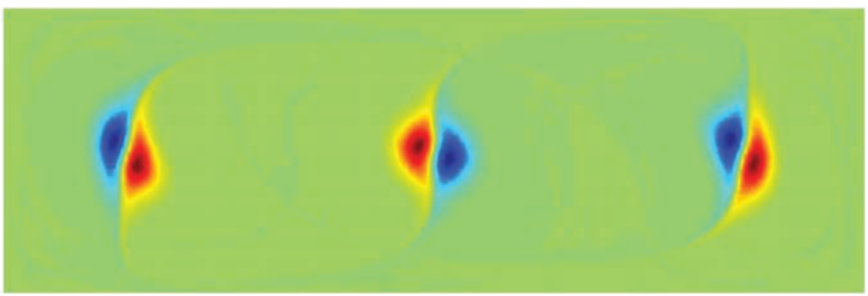
- Return to $\tau_f < 1$ case, where no periodic orbits of low period known
- What are the AISs and ACSs here?

Identifying 'ghost rods': almost-cyclic sets

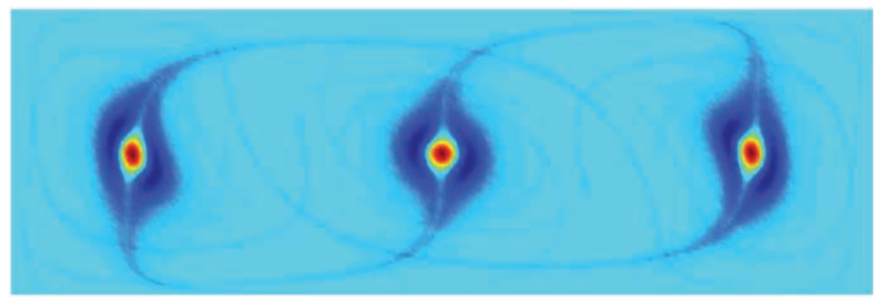
Top eigenvectors of R for $\tau_f = 0.99$ reveal hierarchy of phase space structures



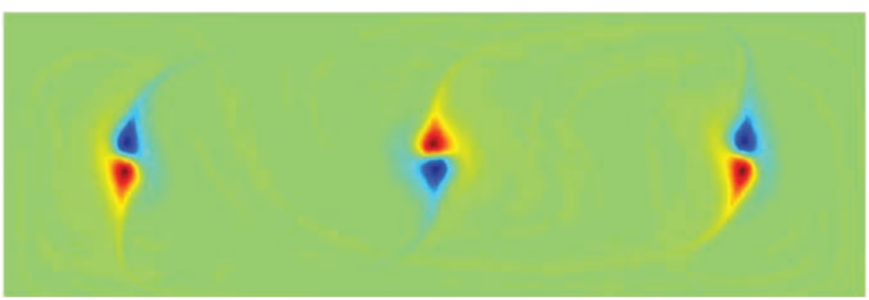
ν_2



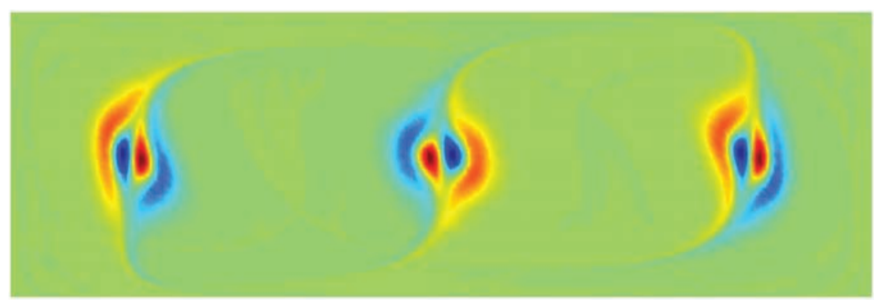
ν_3



ν_4

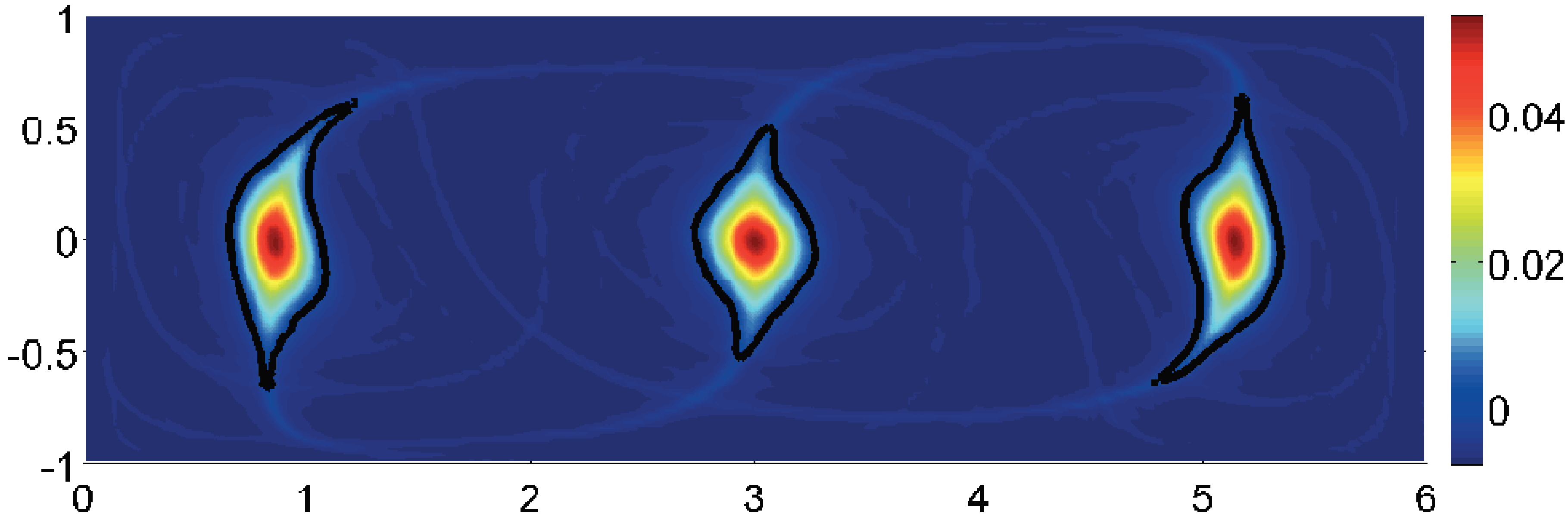


ν_5



ν_6

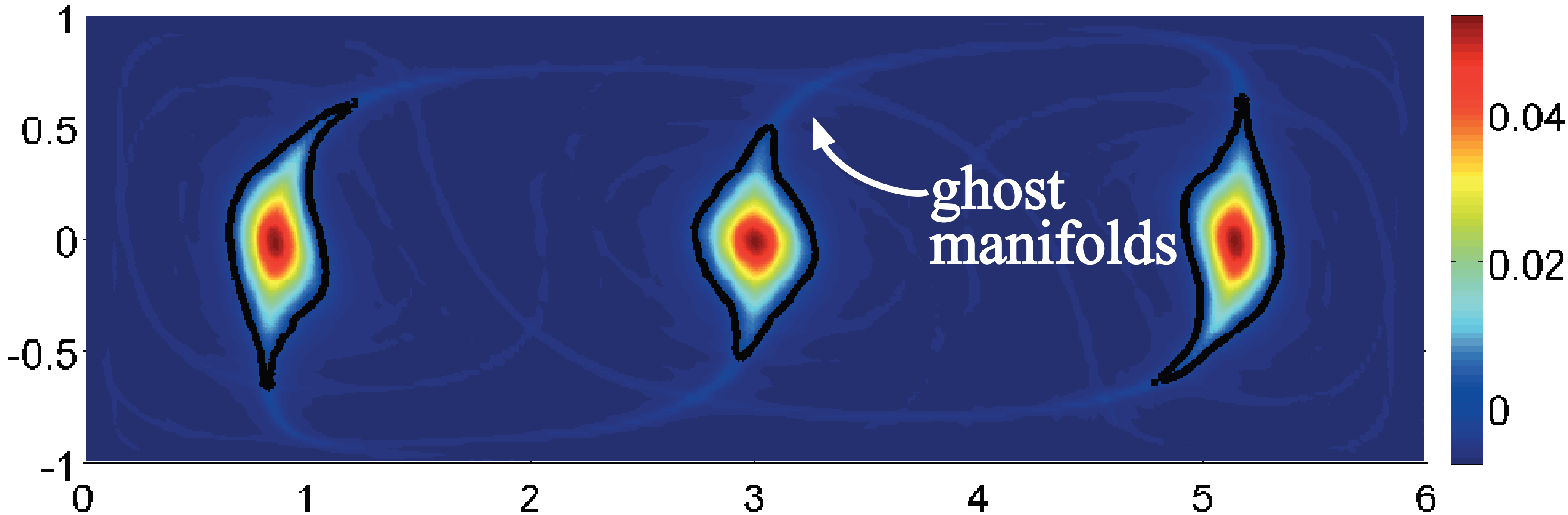
Identifying 'ghost rods': almost-cyclic sets



The zero contour (black) is the boundary between the two almost-invariant sets.

- Three-component AIS made of 3 ACSs of period 3
- ACSs, in effect, replace periodic orbits for TNCT

Identifying ‘ghost rods’: almost-cyclic sets



The zero contour (black) is the boundary between the two almost-invariant sets.

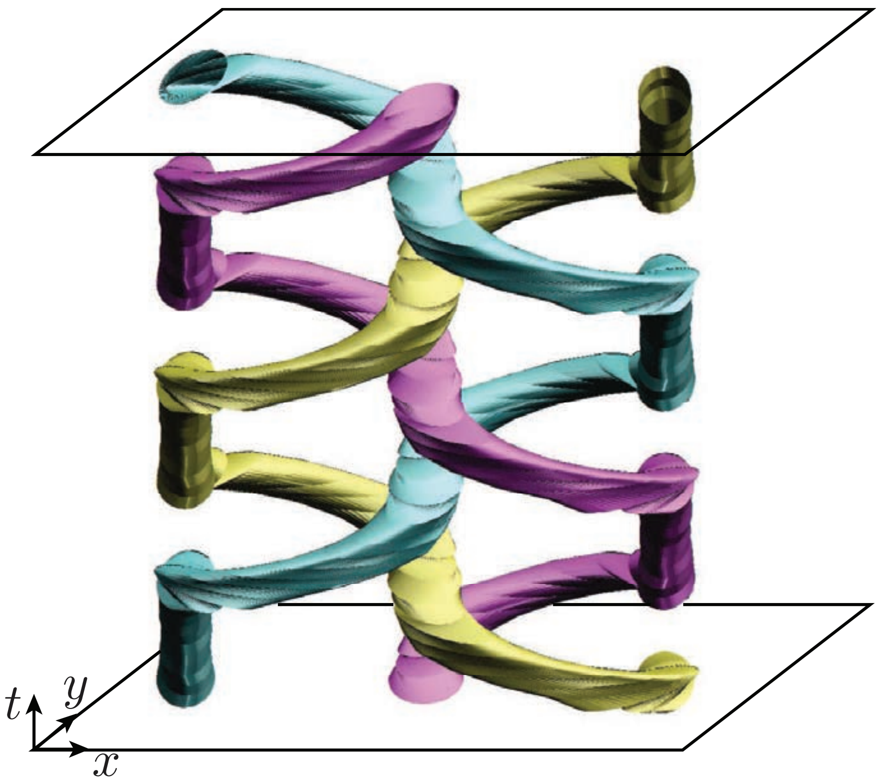
- Three-component AIS made of 3 ACSs of period 3
- ACSs, in effect, replace periodic orbits for TNCT
- Also: we see a **remnant of the ‘stable and unstable manifolds’ of the saddle points**, despite no saddle points – ‘ghost manifolds’?

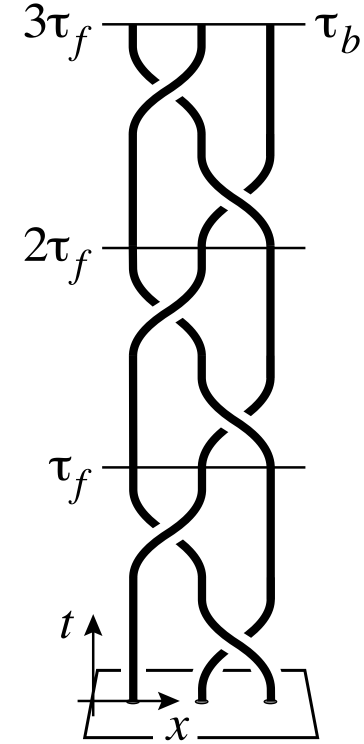
Identifying 'ghost rods': almost-cyclic sets

Almost-cyclic sets stirring the surrounding fluid like 'ghost rods'
— **works even when periodic orbits are absent!**

Movie shown is second eigenvector for $R_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

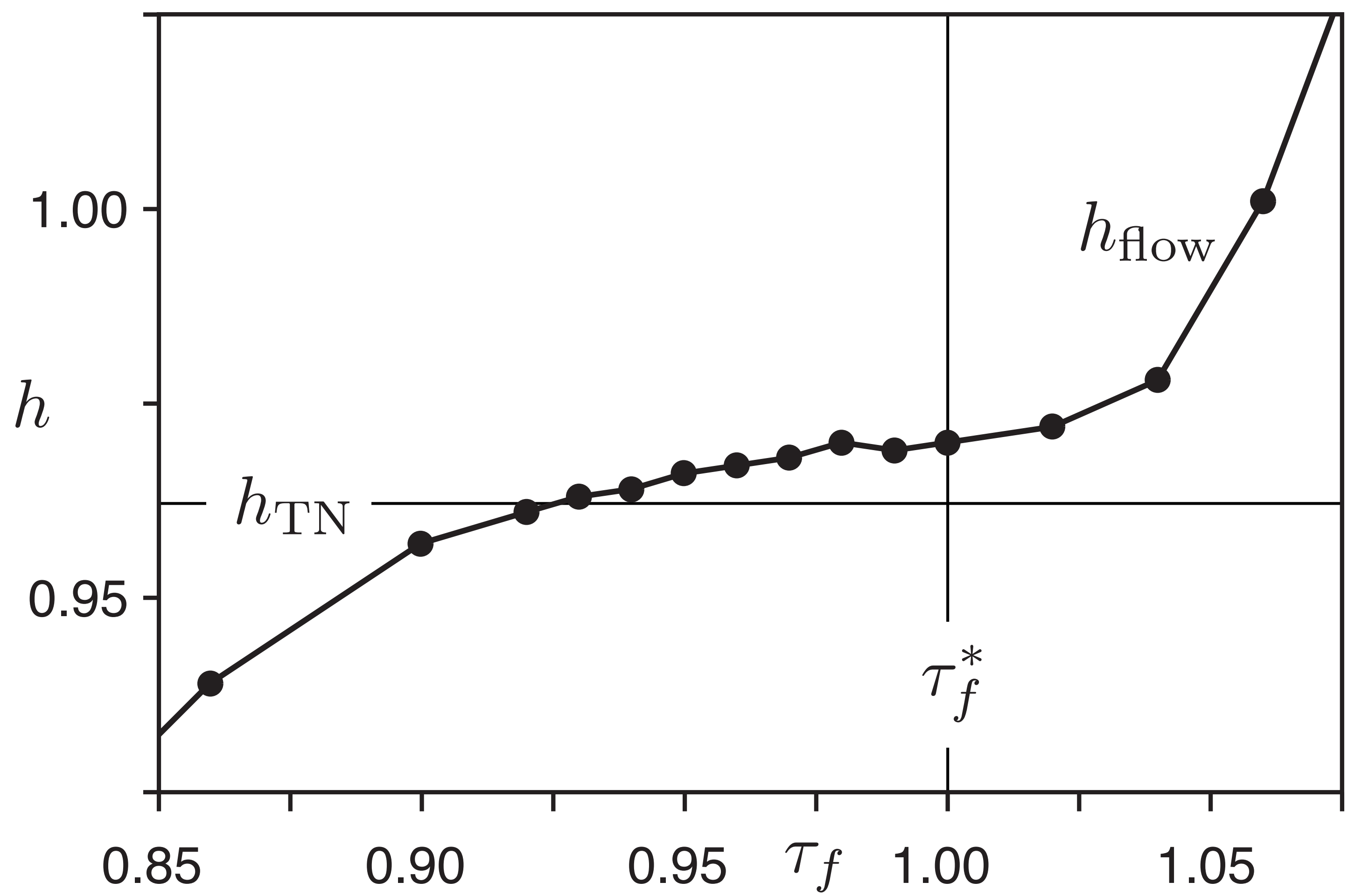
Identifying 'ghost rods': almost-cyclic sets





Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of fluid
— But, theorems apply only to periodic points!
— Stremler, Ross, Grover, Kumar [2011] Phys. Rev. Lett.

Topological entropy vs. bifurcation parameter



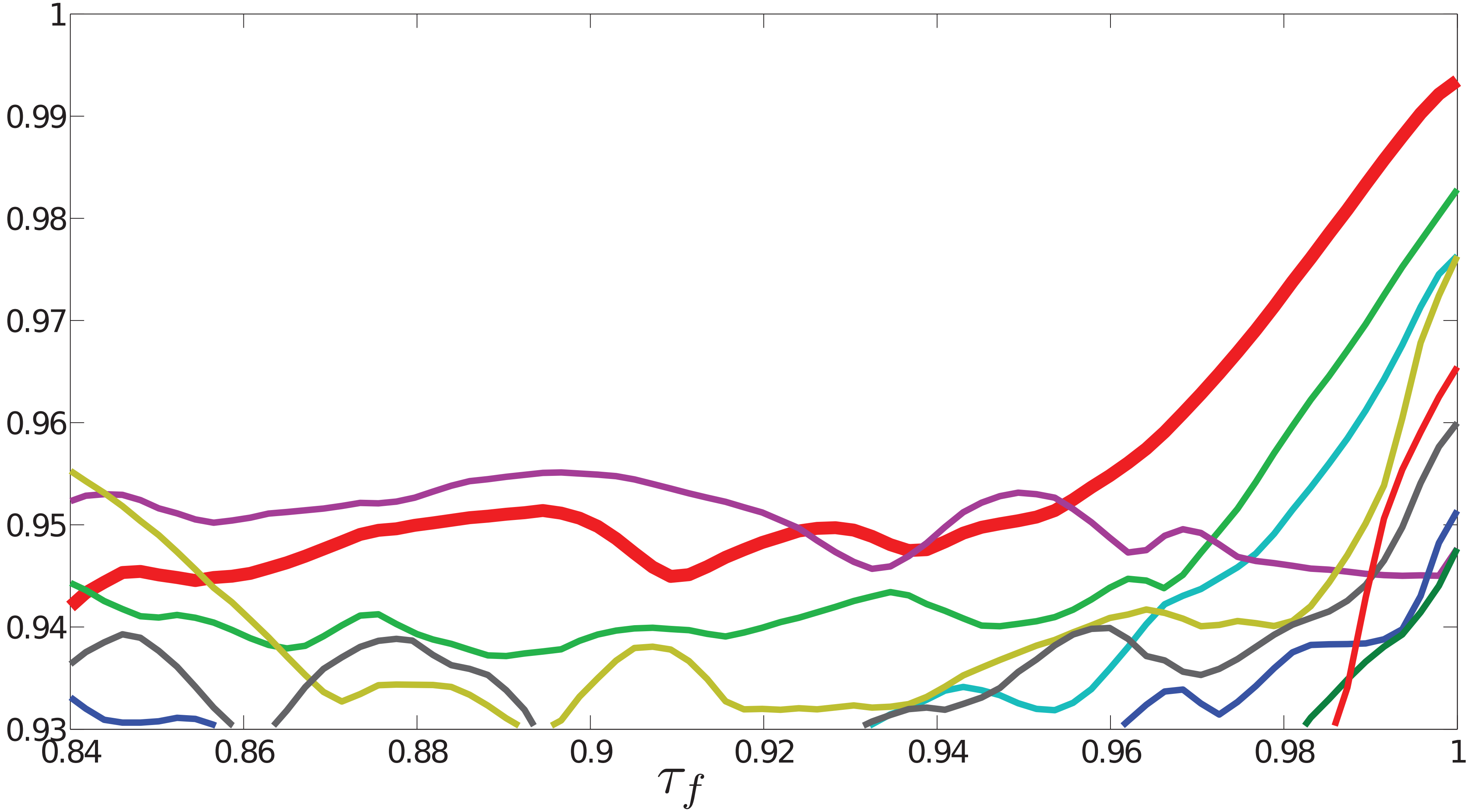
topological entropy as a function of τ_f

- h_{TN} shown for ACS braid on 3 strands

Eigenvalues/eigenvectors vs. bifurcation parameter

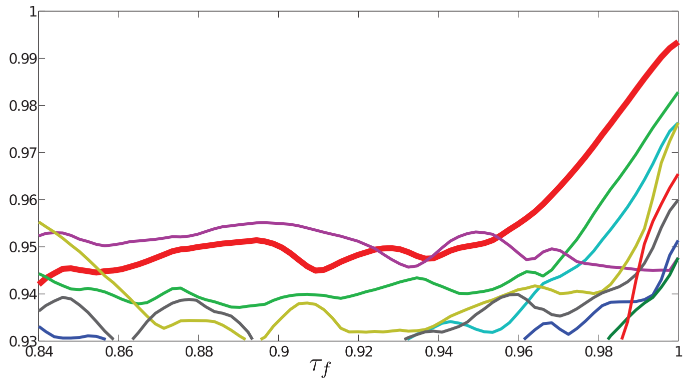
Eigenspectrum of P changes with the parameter τ_f

Eigenvalues/eigenvectors vs. bifurcation parameter



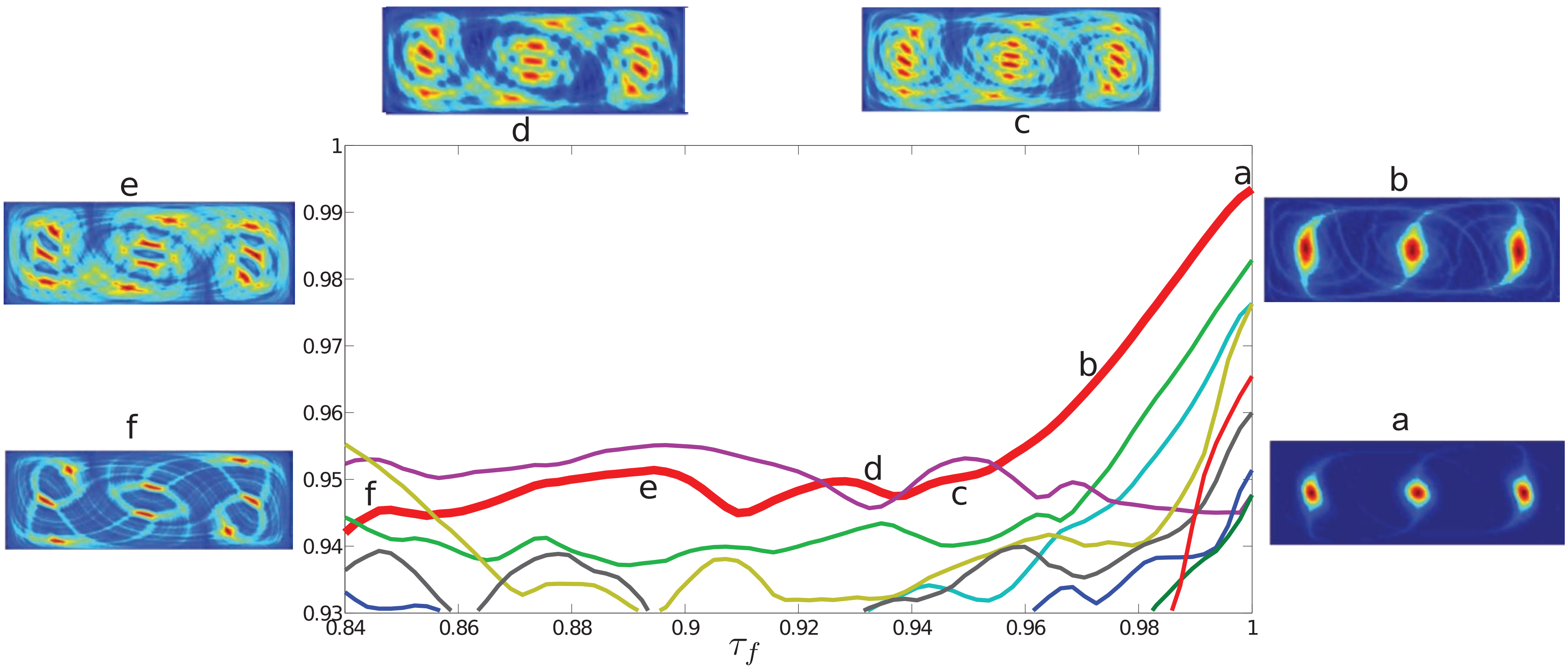
Top eigenvalues of R as parameter τ_f changes

Eigenvalues/eigenvectors vs. bifurcation parameter



Genuine eigenvalue crossings? Eigenvalues generically avoid crossings if there is no symmetry present (Dellnitz, Melbourne, 1994)

Eigenvalues/eigenvectors vs. bifurcation parameter



Movie shows change in eigenvector along thick red branch (a to f), as τ_f decreases.

Grover, Ross, Stremler, Kumar [2012] Chaos

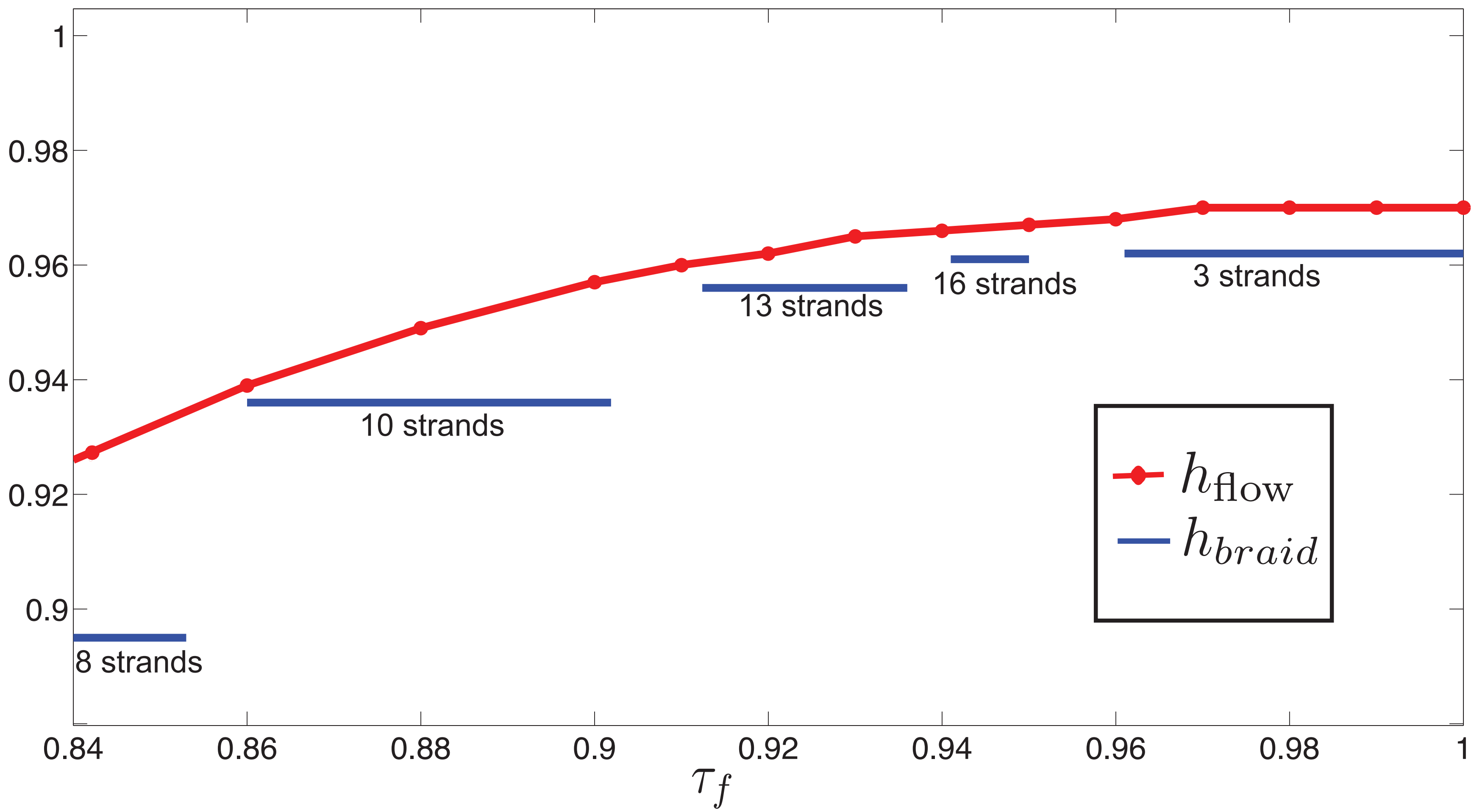
Bifurcation of ACSs

For example, braid on 13 strands for $\tau_f = 0.93$

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Thurson-Nielsen for this braid provides lower bound on topological entropy

Sequence of ACS braids bounds entropy

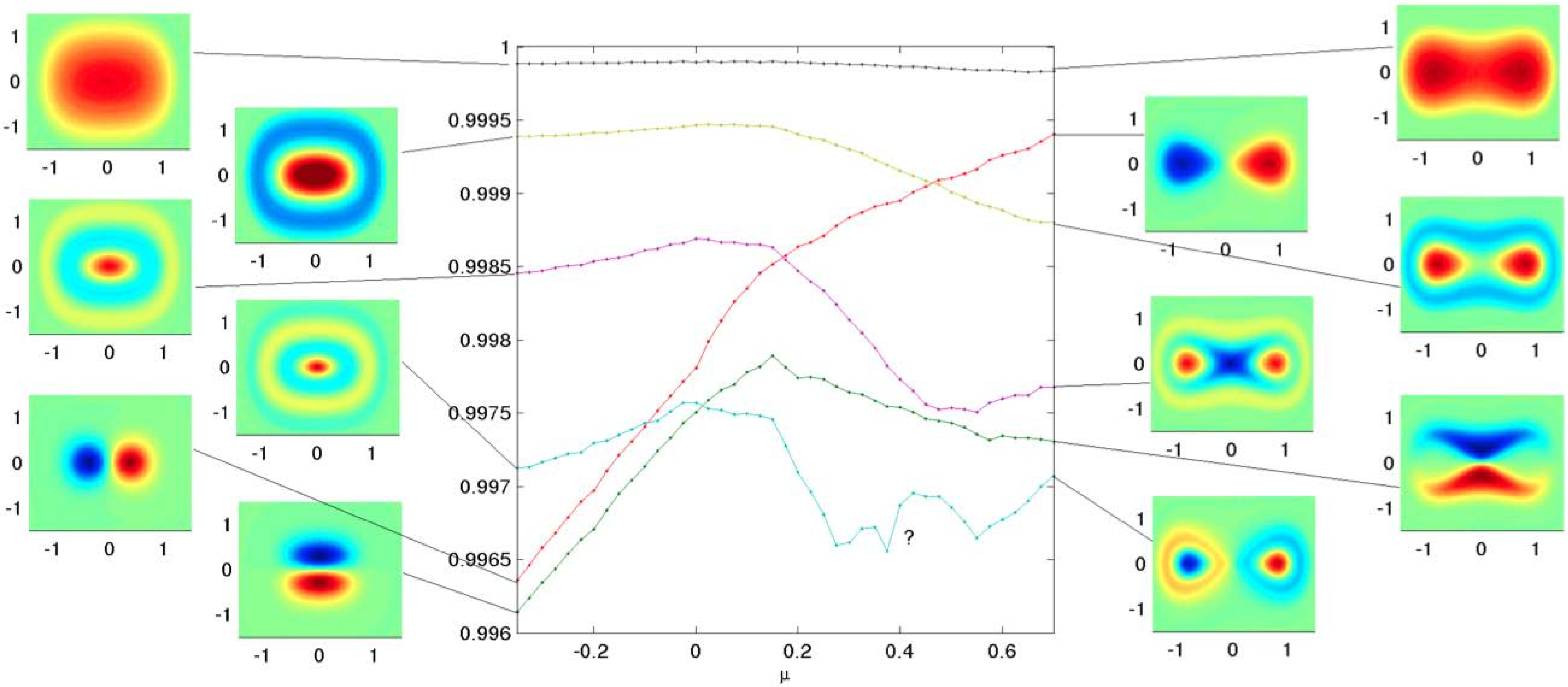


For various braids of ACSs, the calculated entropy is given, bounding from below the true topological entropy over the range where the braid exists

Grover, Ross, Stremler, Kumar [2012] Chaos

Speculation: trends in eigenvalues/modes for prediction

Speculation: trends in eigenvalues/modes for prediction

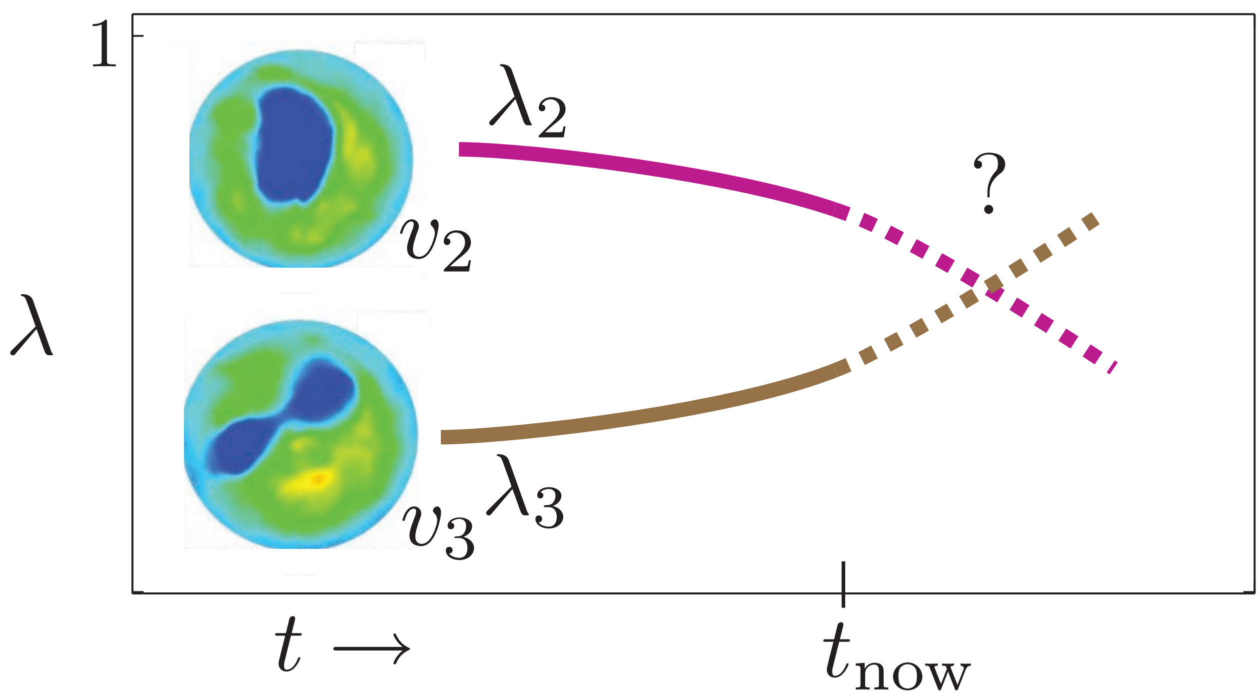


- Duffing system with small noise: six largest eigenvalues of the reversibilized discretized transfer operator in dependence of the bifurcation parameter (Junge, Marsden, Mezic 2004)

Predict critical transitions in geophysical transport?

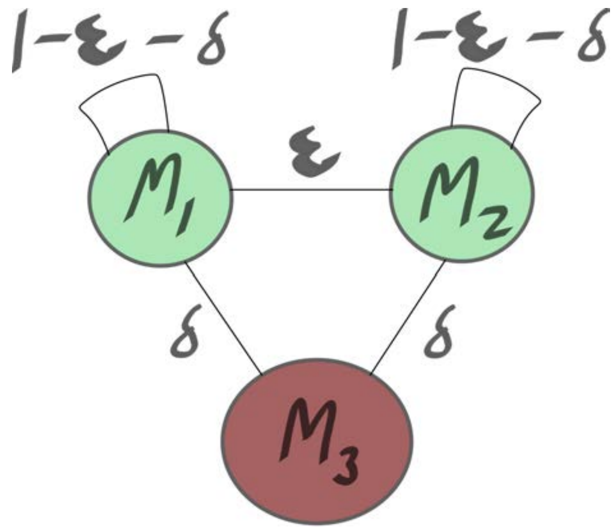
Ozone data (Lekien and Ross [2010] Chaos)

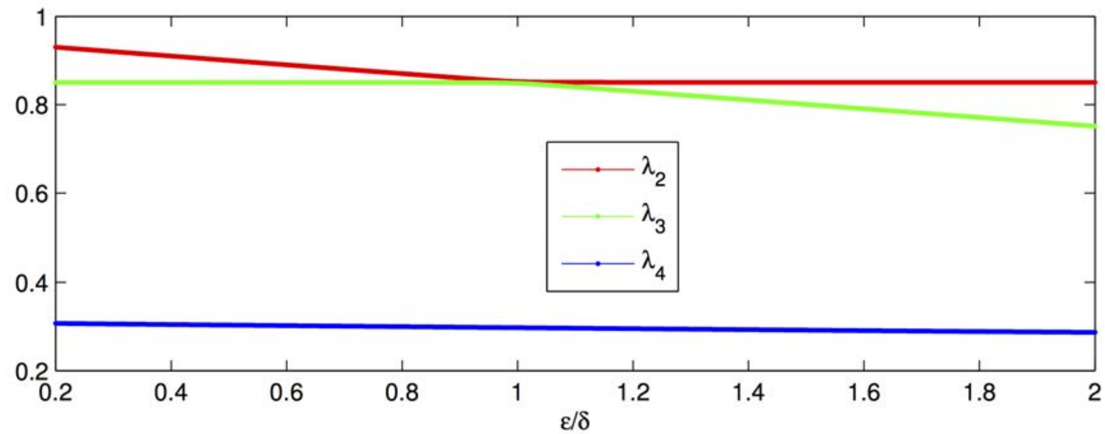
Predict critical transitions in geophysical transport?



- Different eigenmodes can correspond to dramatically different behavior.
- Some eigenmodes increase in importance while others decrease
- Can we predict dramatic changes in system behavior?
- e.g., predicting major changes in geophysical transport patterns??
- Ongoing work with E. Bollt, O. Junge, K. Padberg-Gehle, N. Santitissadeekorn

Predict critical transitions in geophysical transport?

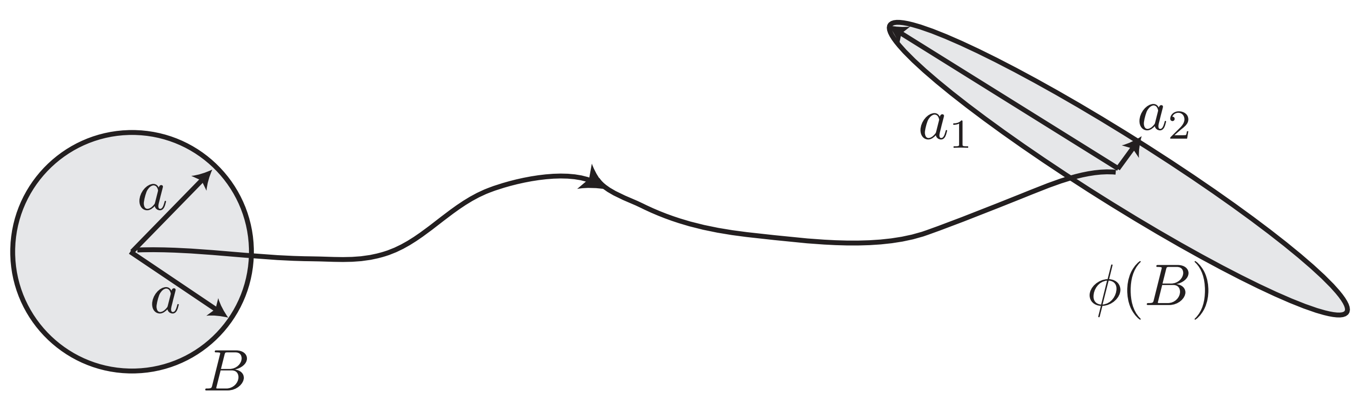




- Look at simplest representatives of **mode-switching** or other bifurcation phenomena

Coherent sets and set-based definition of FTLE

- Consider, e.g., a flow ϕ_t^{t+T} in $(x_1, x_2) \in \mathbb{R}^2$.
- Evolution of set $B \subset \mathbb{R}^2$ viewed as evolution of two random variables X_1 and X_2 with joint probability density function $f(x_1, x_2)$, initially uniform on B, $f = \frac{1}{\mu(B)} \mathbb{1}_B$ ($\mathbb{1}_B$ the characteristic function of B)
- Under the action of the flow ϕ_t^{t+T}, f is mapped to $\mathcal{P}f$ where $\mathcal{P}$ is the associated Perron-Frobenius operator.
- Let $I(f)$ be the covariance of f and $I(\mathcal{P}f)$ the covariance of $\mathcal{P}f$.



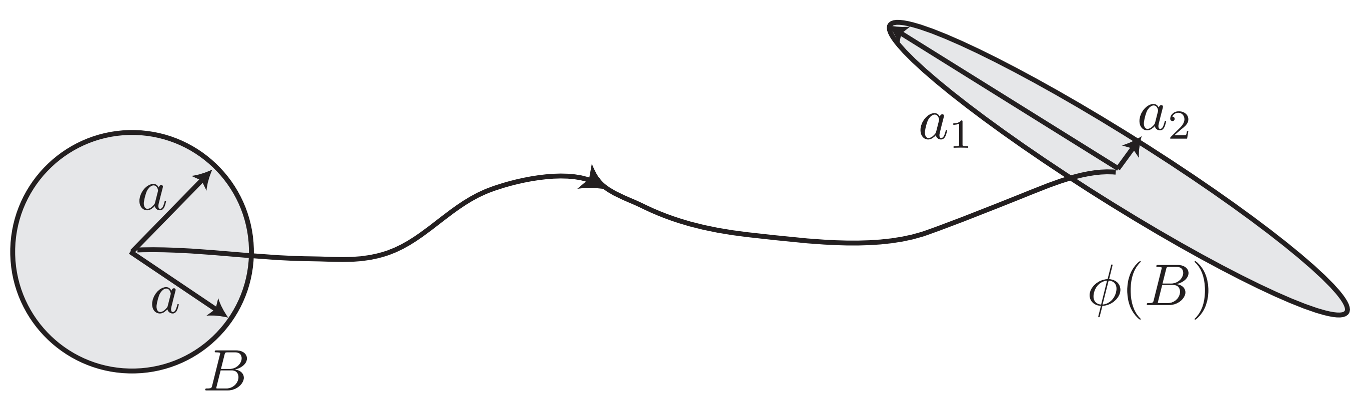
Deformation of a disk under the flow during $[t, t+T]$

Coherent sets and set-based definition of FTLE

- **Definition.** The **covariance-based FTLE** of B is

$$\sigma_I(B,t,T) = \frac{1}{|T|}\log\left(\sqrt{\frac{\lambda_{\max}(I(\mathcal{P}f))}{\lambda_{\max}(I(f))}}\right)$$

- Tallapragada & Ross [2013] *Comm. Nonlinear Sci. Numerical Simulation*
- Reduces to usual definition of FTLE, σ, in the limit of small sets B; e.g., for the disk in an area-perserving flow, $\sigma = \sigma_I = \frac{1}{|T|}\log\left(\frac{a_1}{a}\right)$

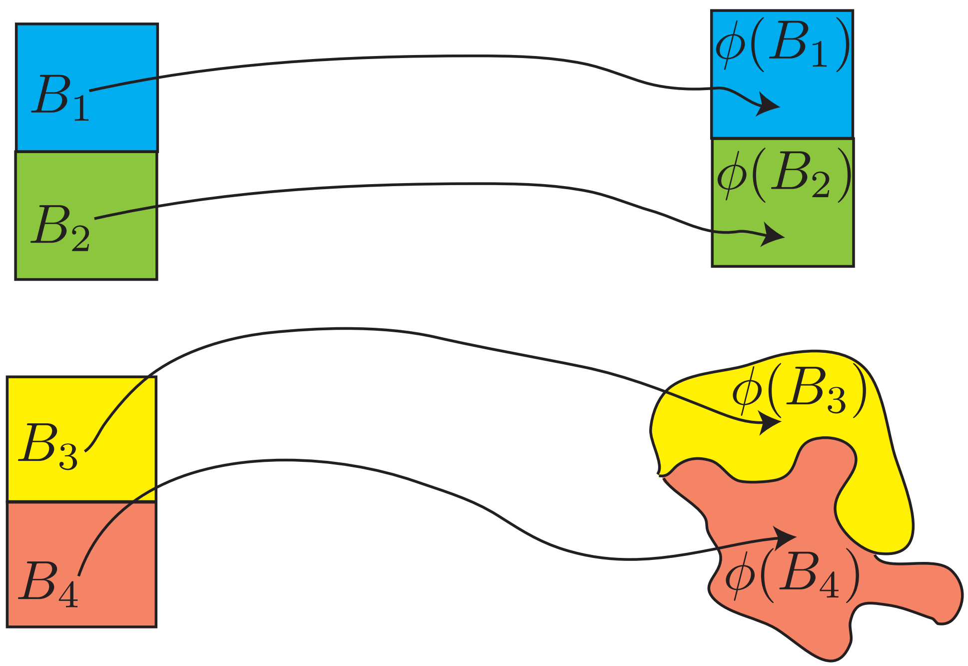


Deformation of a disk under the flow during $[t, t+T]$

Coherent sets and set-based definition of FTLE

- The coherence of a set B during $[t, t+T]$ is measured by closeness of $\sigma_I(B, t, T)$ to zero.

- Essential feature of a coherent set: scalar dispersion within it is low.
- This definition also can identify non-mixing **translating** sets.



Coherent sets and set-based definition of FTLE

- The coherence of a set B during $[t, t+T]$ is measured by closeness of $\sigma_I(B, t, T)$ to zero.

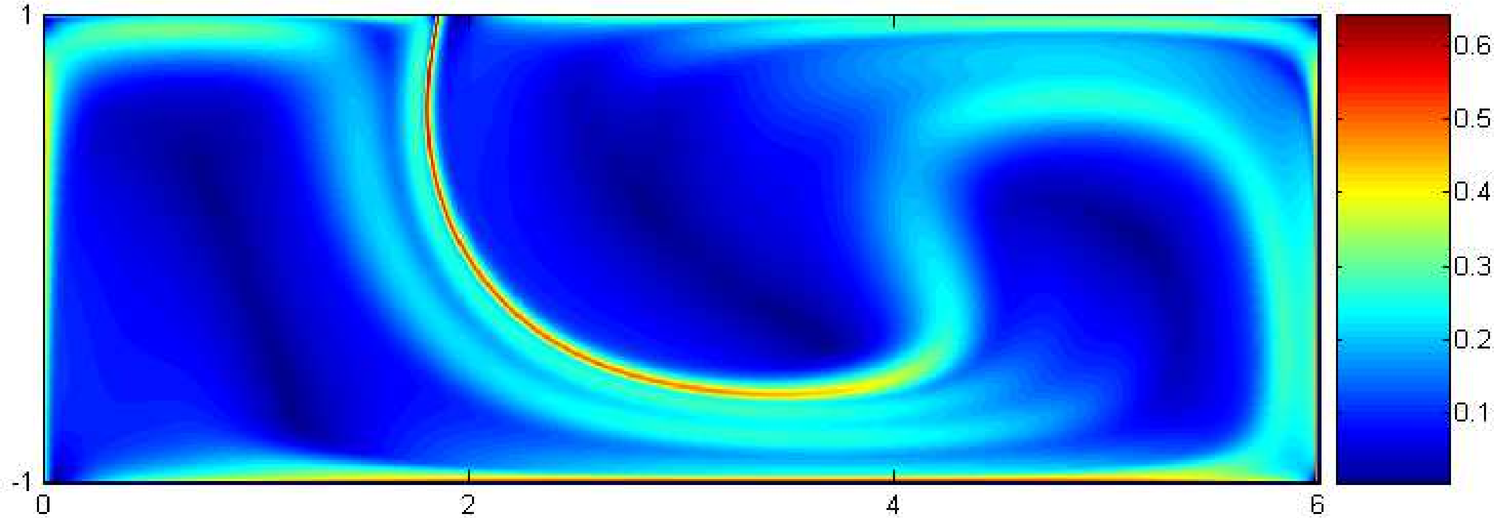
- Essential feature of a coherent set: scalar dispersion within it is low.
- This definition also can identify non-mixing **translating** sets.

- Preselection: Set a heuristic threshold on $\sigma_I(B, t, T)$ to identify regionz which may contain coherent sets.
- Then use other methods to identify optimal coherence.

  e.g., Froyland, Santitisadeekorn, Monahan [2010], Haller, Beron-Vera [2012]

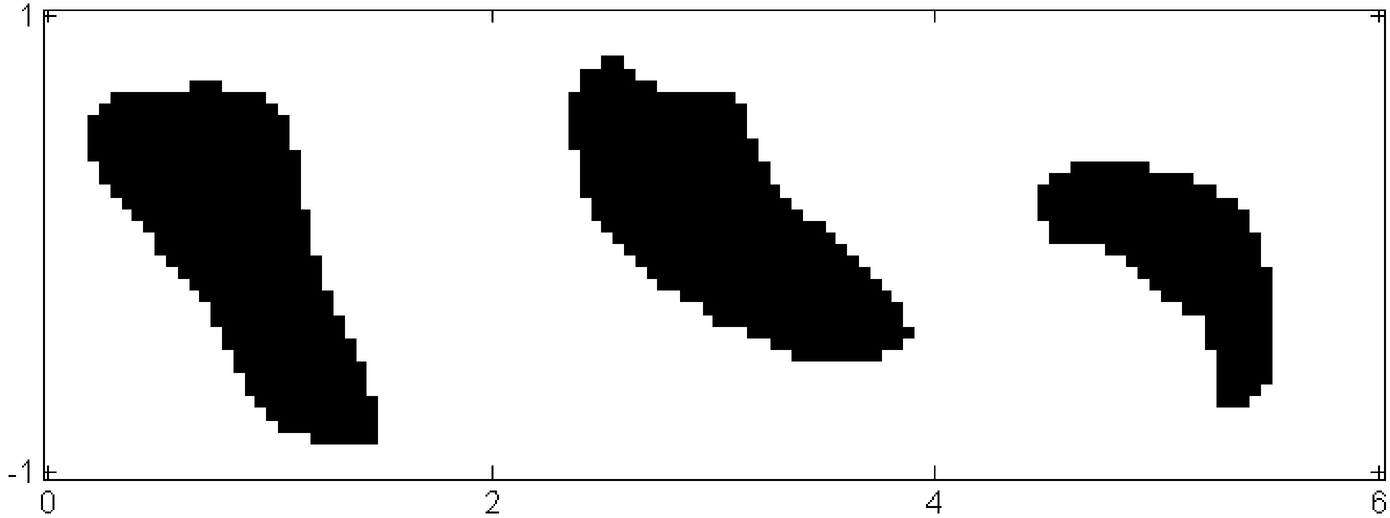
- Notice, coherent sets will valleys be separated by ridges of high FTLE, i.e., LCS

Coherent sets in microfluidic mixer from before



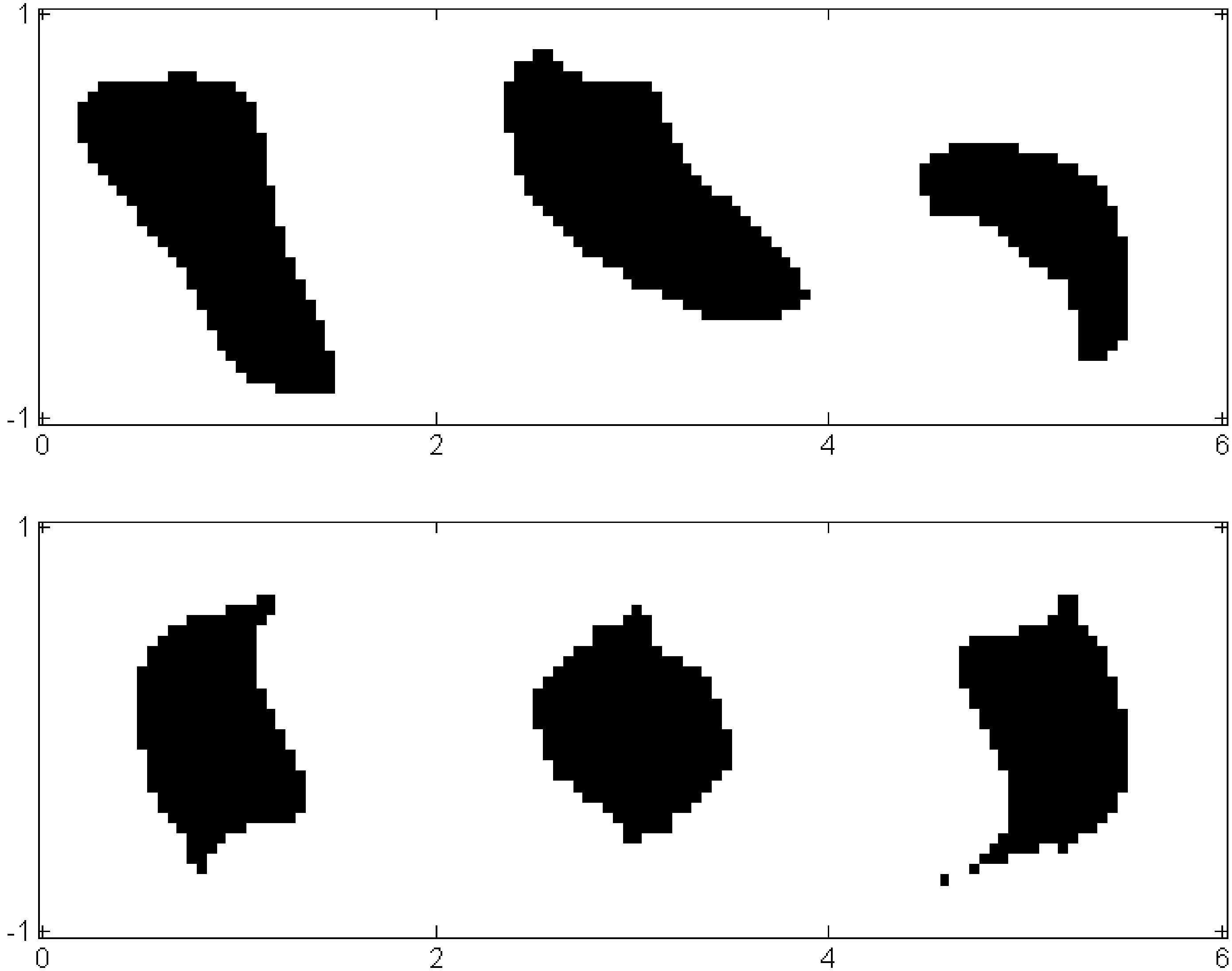
FTLE during $[0, \tau_f]$

Coherent sets in microfluidic mixer from before



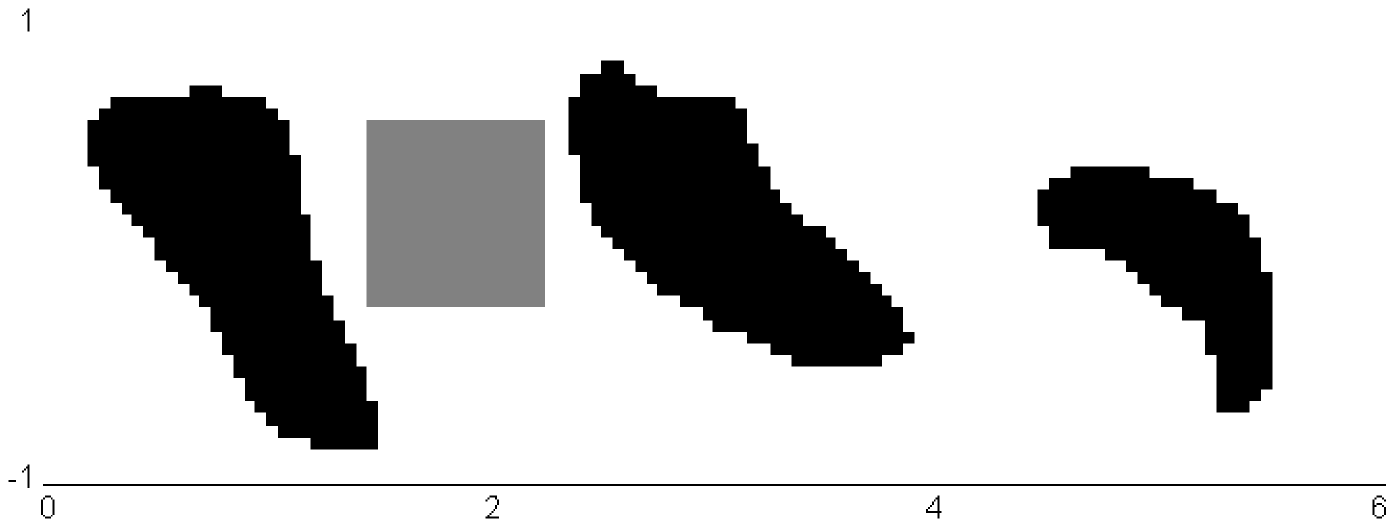
Sets of coherence $\sigma_I(0, \tau_f) < 0.06$

Coherent sets in microfluidic mixer from before



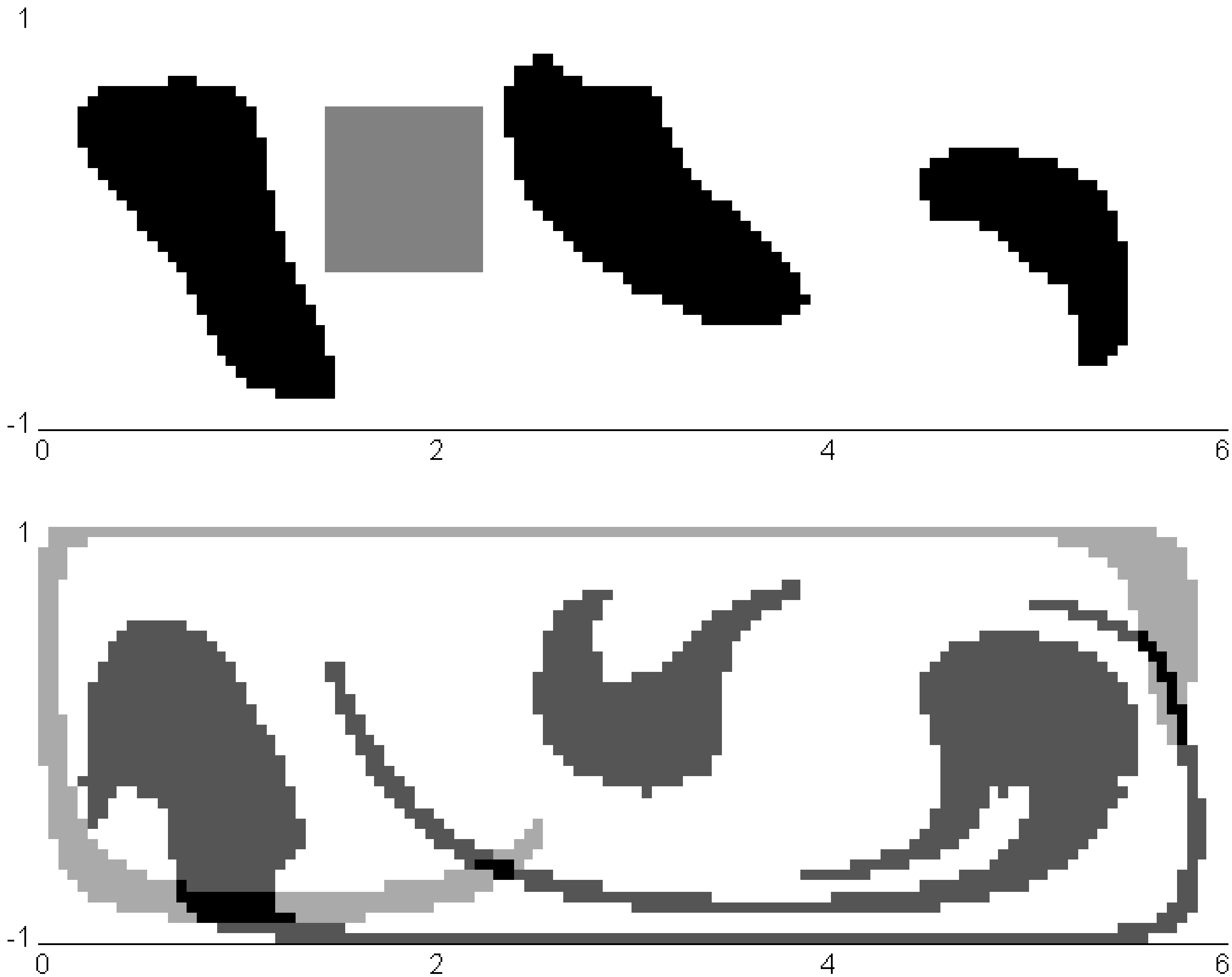
Compare with AIS (from second eigenvector of R)

Coherent sets in microfluidic mixer from before



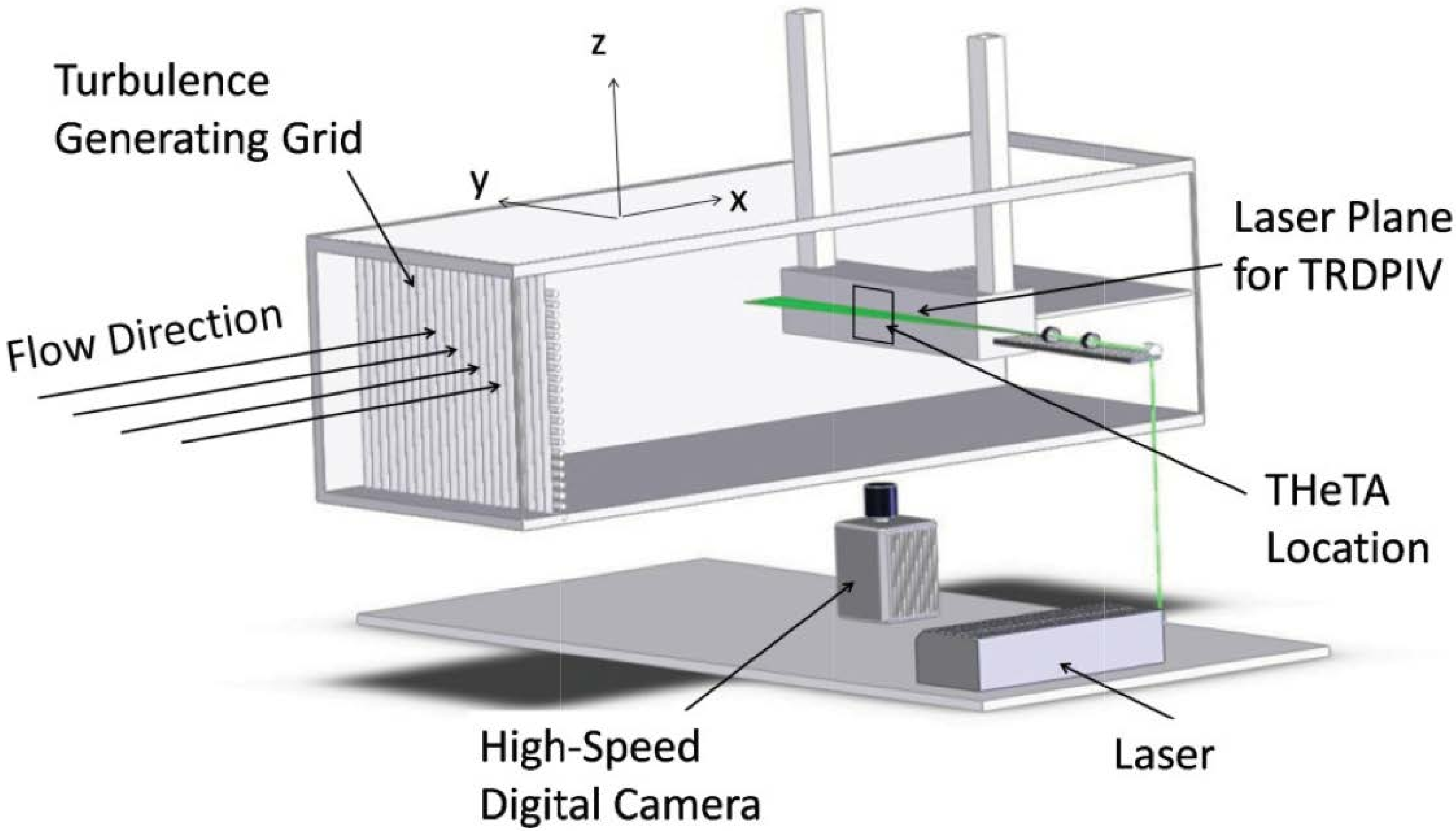
Compare high-coherence sets with low-coherence set (gray)

Coherent sets in microfluidic mixer from before



Coherent sets in fluid experiments

A particle image velocimetry (PIV) fluid experiment (Hubble [2011]); Vlachos lab (Virginia Tech/Purdue)



Data processing and FTLE computations by S. Raben, 2012

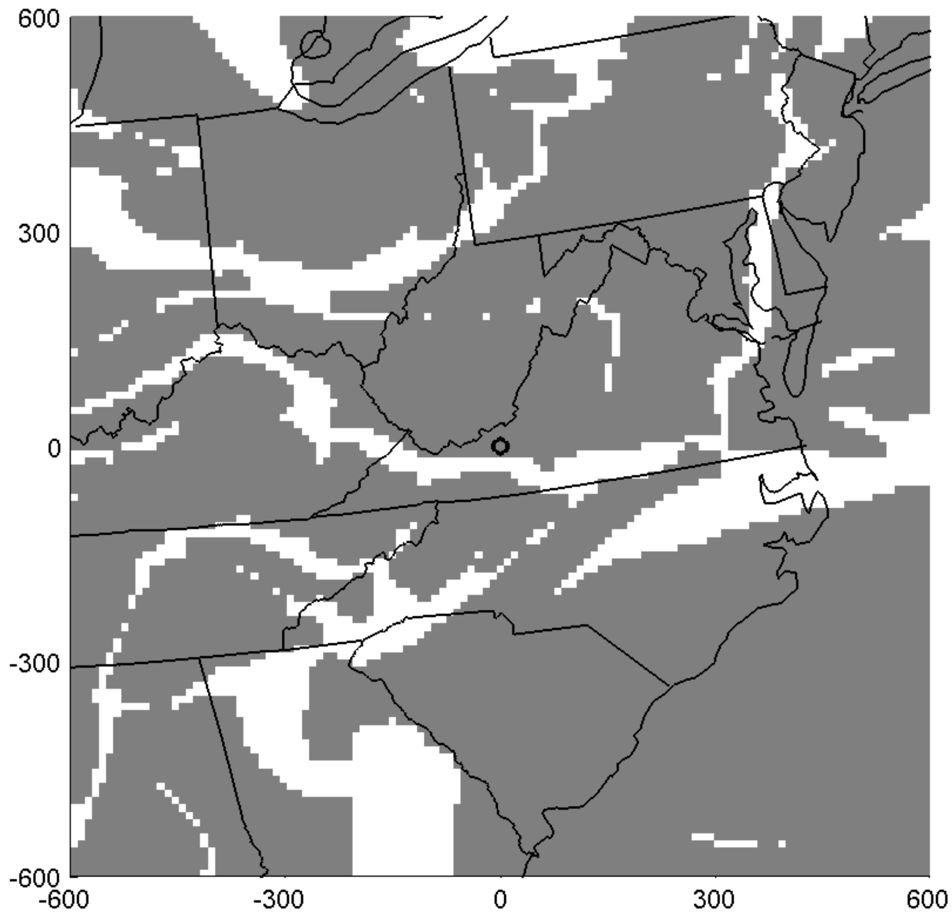
Coherent sets in fluid experiments

Coherent sets in forward time $[0, 1\ \mathrm{sec}]$ along with usual FTLE ridges

Coherent sets in fluid experiments

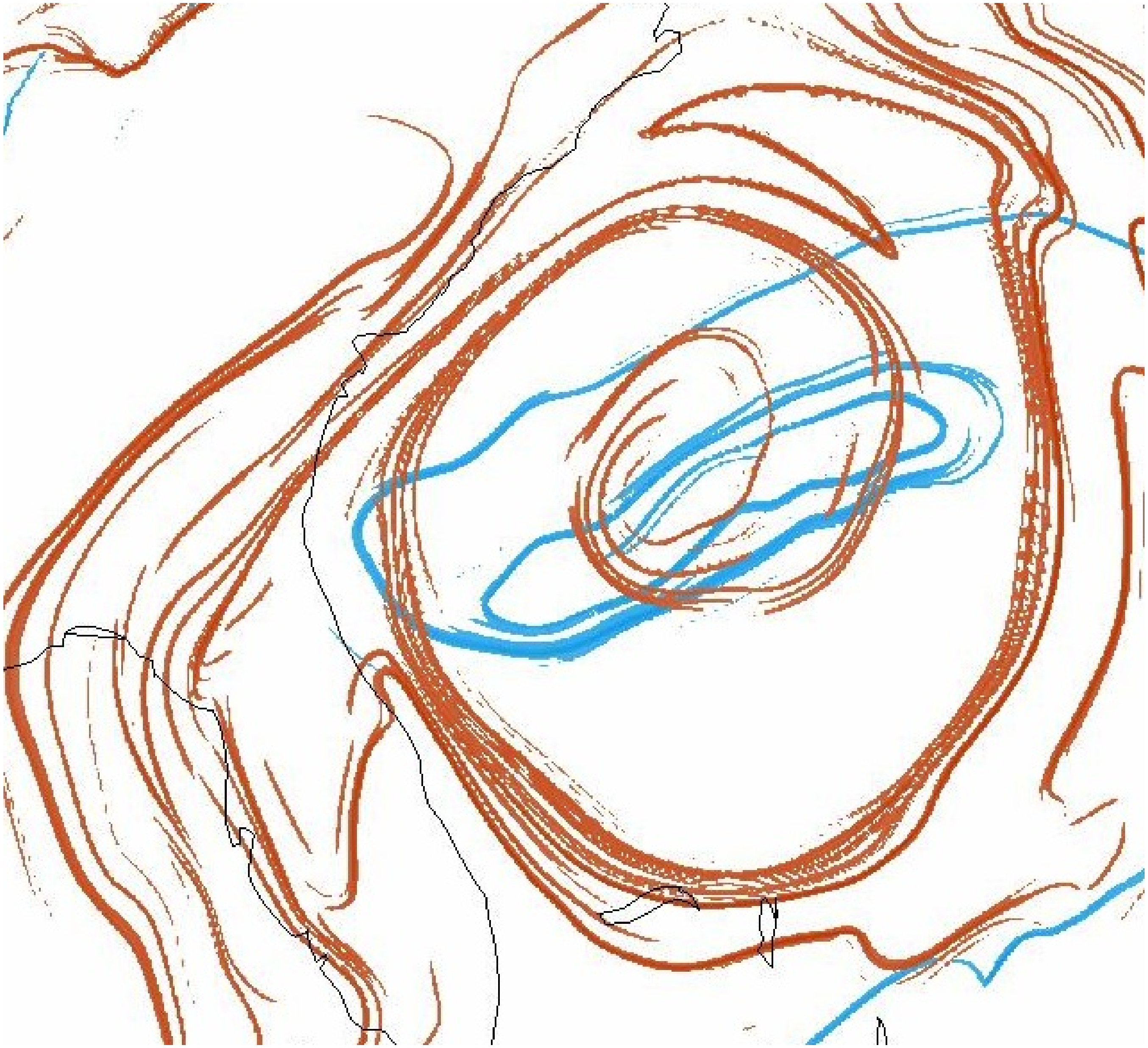
Coherent sets in forward and backward time, $[-1\ \mathrm{sec}, 1\ \mathrm{sec}]$

Coherent sets in the atmosphere



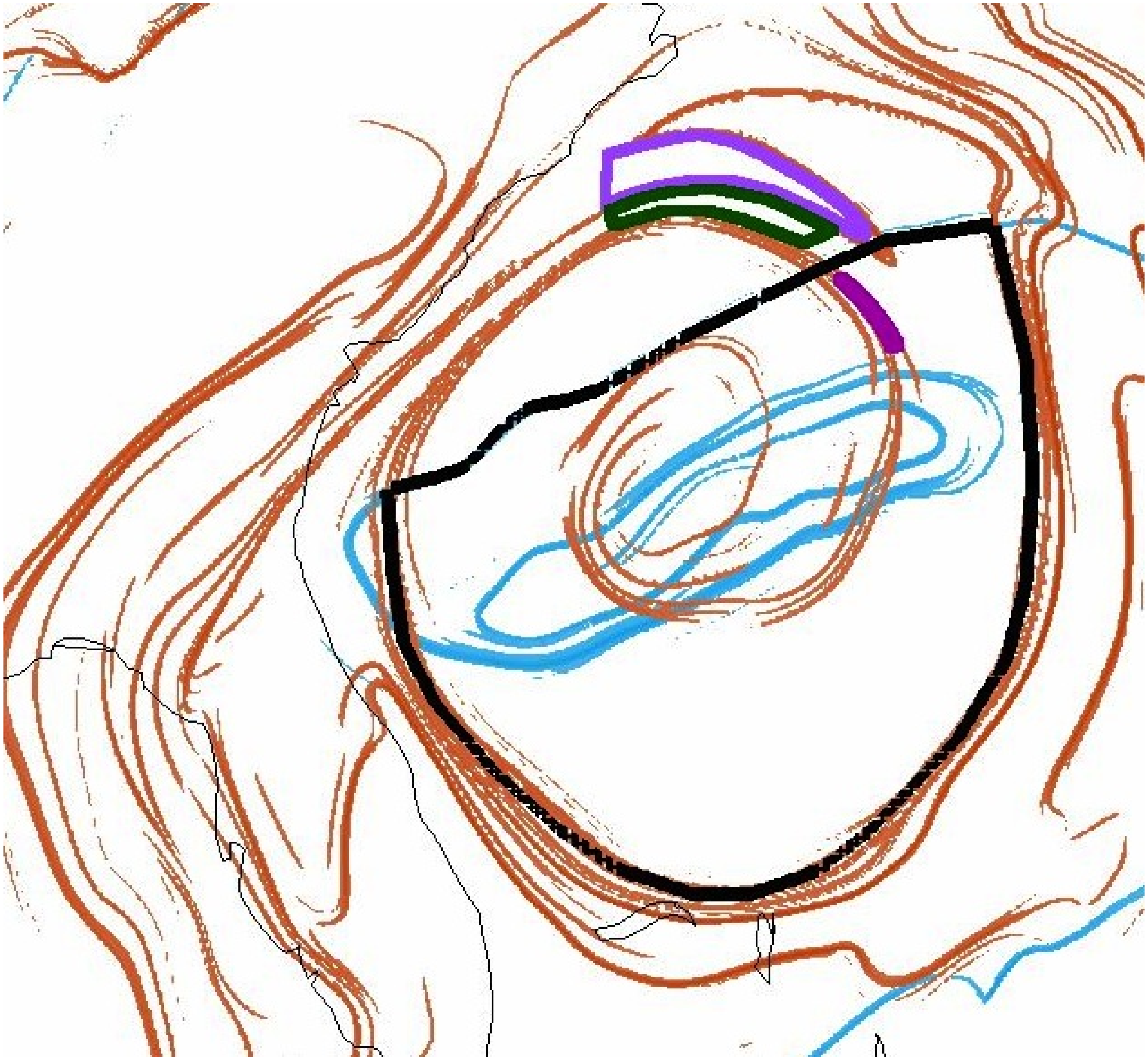
- Coherent sets during 24 hours starting 09:00 1 May 2007

Coherent sets in the atmosphere that braid



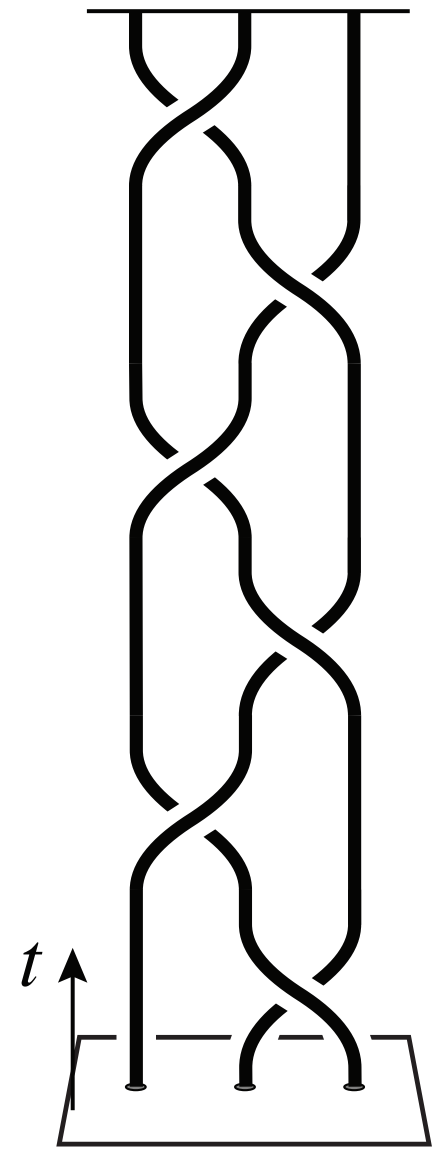
Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)

Coherent sets in the atmosphere that braid



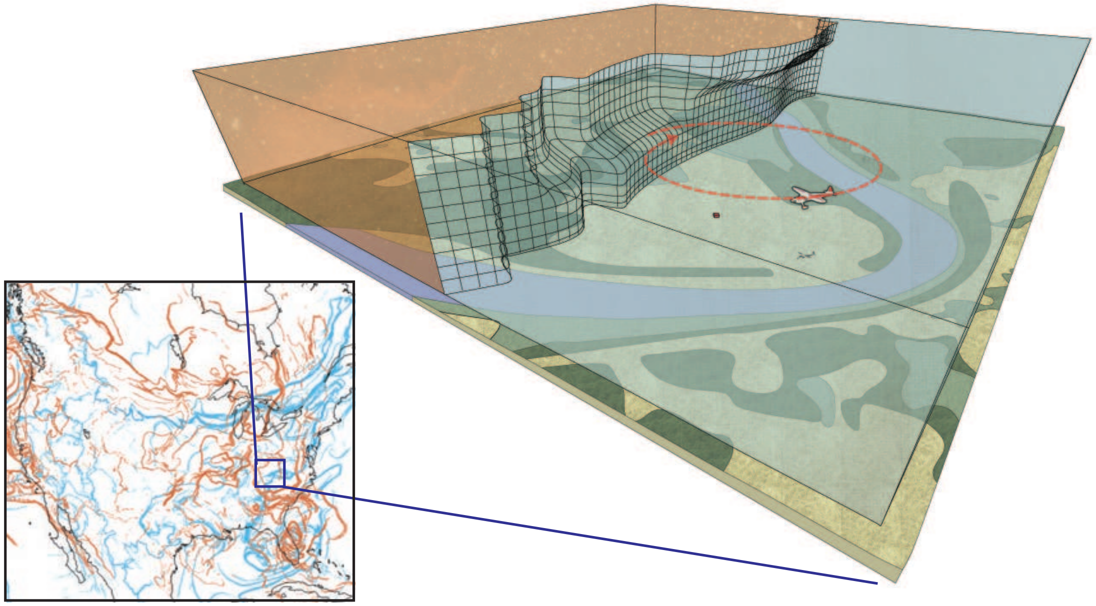
three sets: magenta, green, purple

Coherent sets in the atmosphere that braid

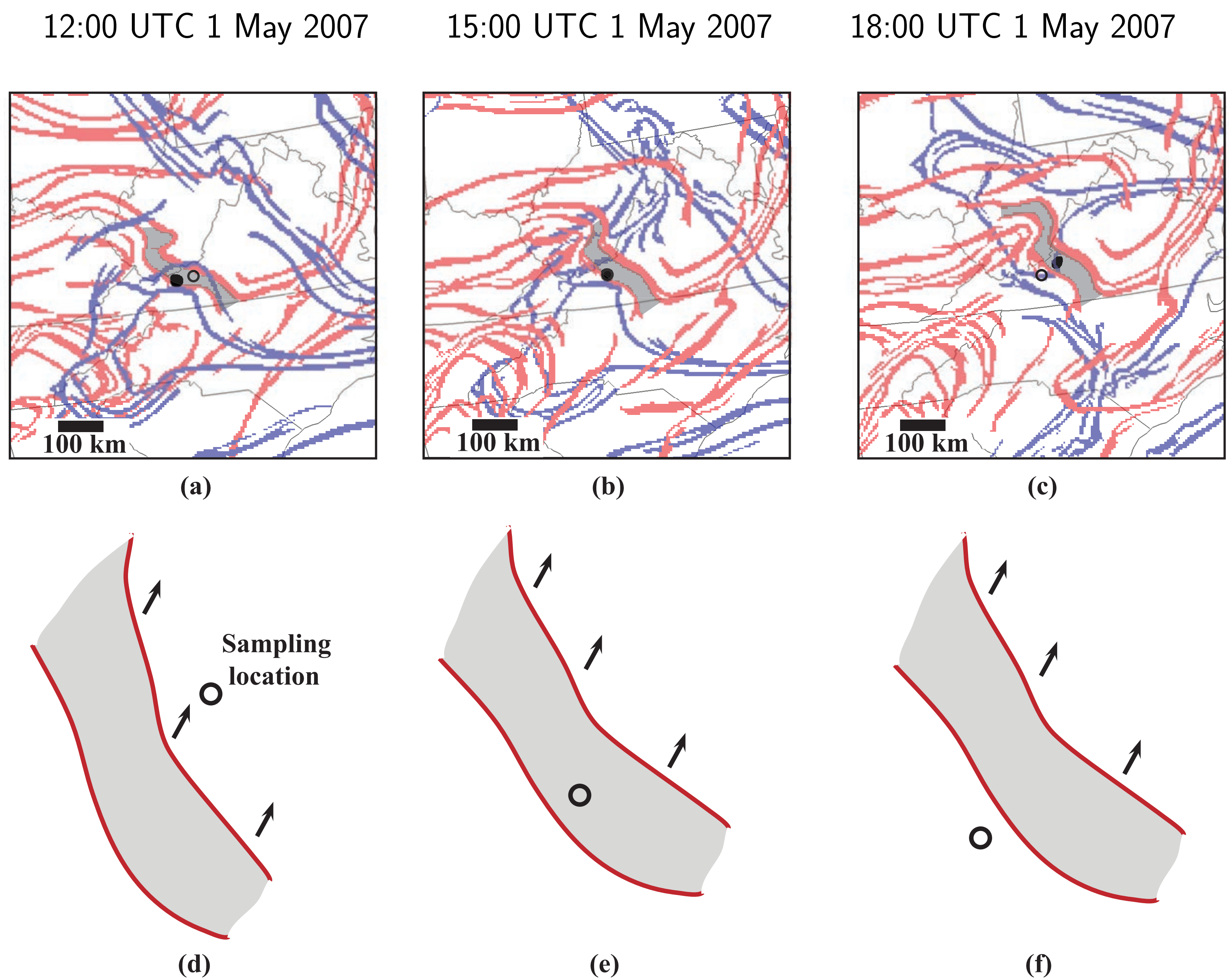


Sets form pseudo-Anosov braid on three strands

Airborne diseases which ride coherent sets



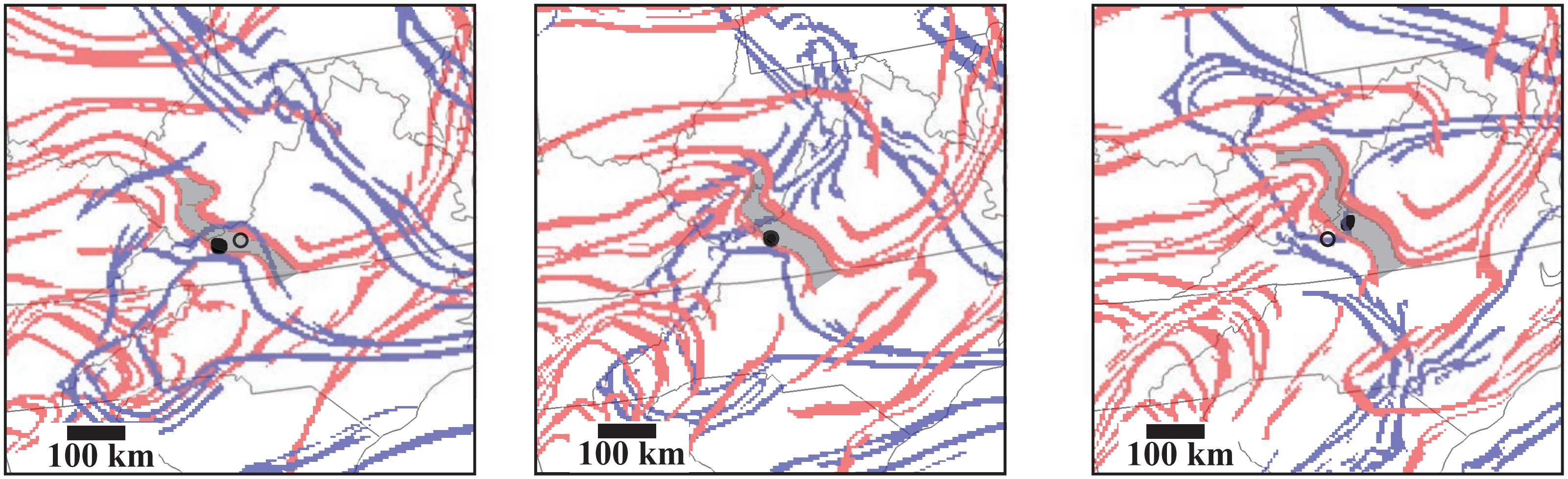
Coherent filament with high pathogen values



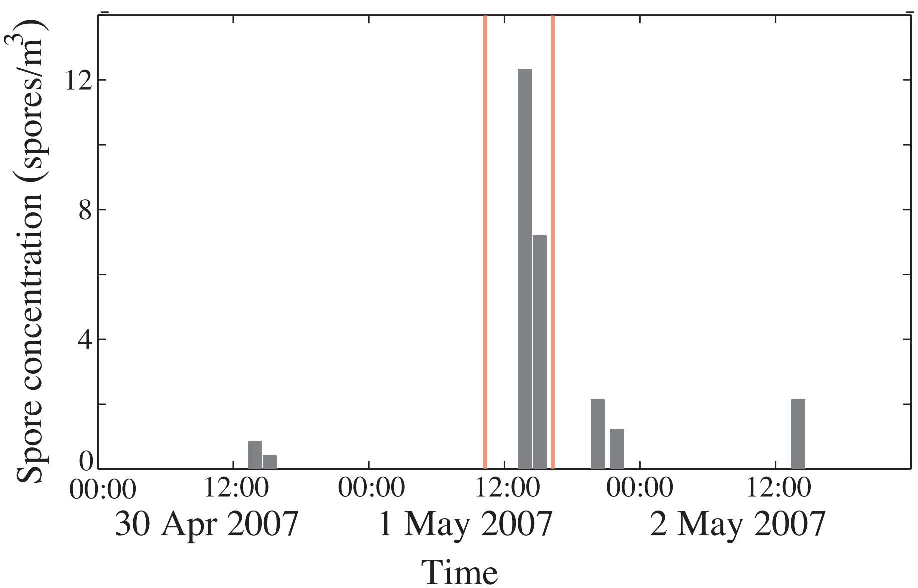
Tallapragada et al [2011] Chaos; Schmale et al [2012] Aerobiologia; BozorgMagham et al [2013] Physica D

Coherent filament with high pathogen values

12:00 UTC 1 May 2007 15:00 UTC 1 May 2007 18:00 UTC 1 May 2007



(a) (b) (c)



Tallapragada et al [2011] Chaos; Schmale et al [2012] Aerobiologia; BozorgMagham et al [2013] Physica D

Final words on coherent sets from data

- Geophysical and engineering fluid applications of set-oriented approaches
- From FTLE, get first-order picture of coherent sets, the ‘valleys’ as opposed to the ridges; useful for engineers.
- Considered the dependence of the transfer operator spectrum on a system parameter; micro mixer application
  — observed bifurcation of braid along certain branch
  — mode-switching
- Future work: predicting bifurcations in transport structure from transfer operator trends

The End Thank You

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Main Papers:

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