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Control & Dynamical Systems

A Statistical Theory of Transition and Collision Probabilities

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Overview

- *Motivation*
  - Planetary science
  - Comet motion, collisions
  - Chaotic dynamics, intermittency
- *Transport theory*
  - Restricted three-body problem
  - Predictions of theory
  - Comparison with observational data?

Motivation

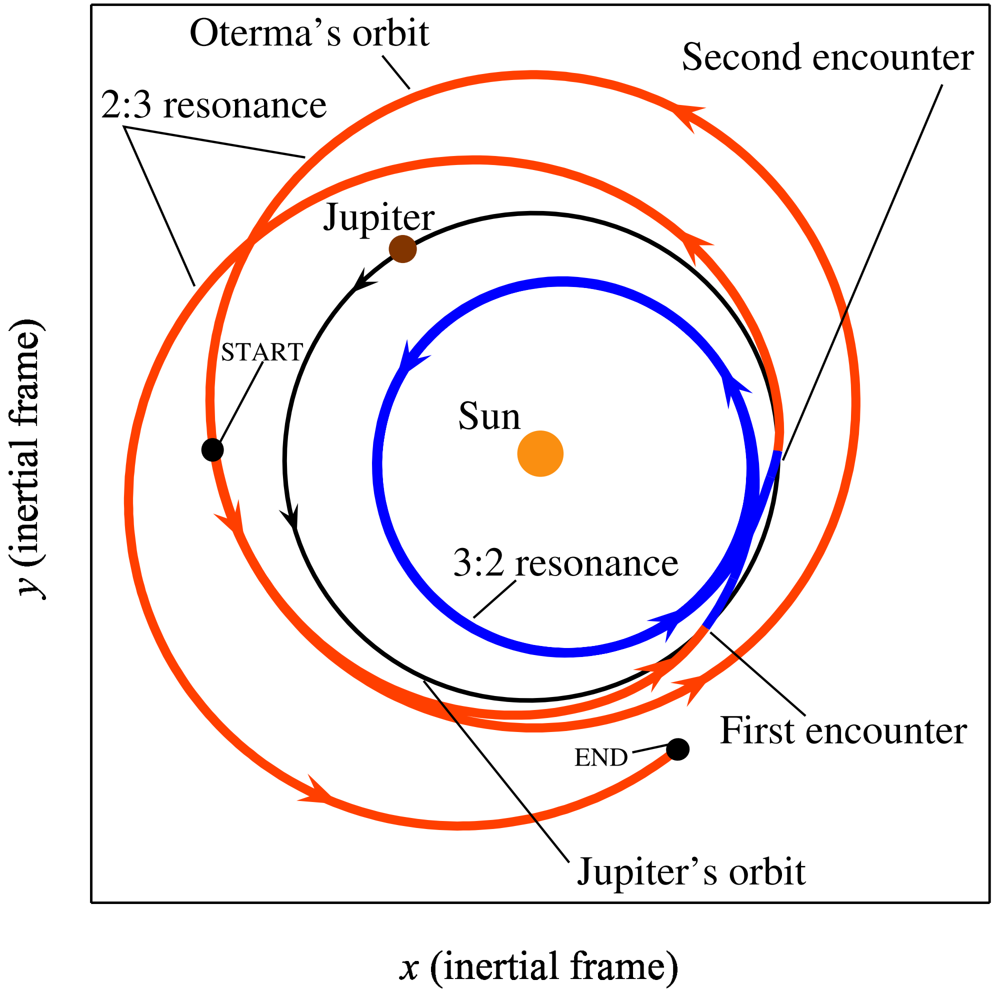
- ***Planetary science***
  - Current astrophysical interest in understanding the transport and origin of solar system material:
    - How likely is **Oterma**-like resonance transition?
    - How likely is **Shoemaker-Levy 9**-type collision with Jupiter? – or an **asteroid collison** with Earth?
    - How does impact ejecta get from Mars to Earth?
  - **Statistical methods** applied to the three-body problem may provide a first-order answer.
  - The "interaction" of several three-body systems increases the complexity.

Jupiter Family Comets

- *Physical example of intermittency*
  - We consider the **historical record** of the comet **Oterma** from 1910 to 1980
    - first in an inertial frame
    - then in a rotating frame
    - a special case of pattern evocation
  - Similar pictures exist for many other comets

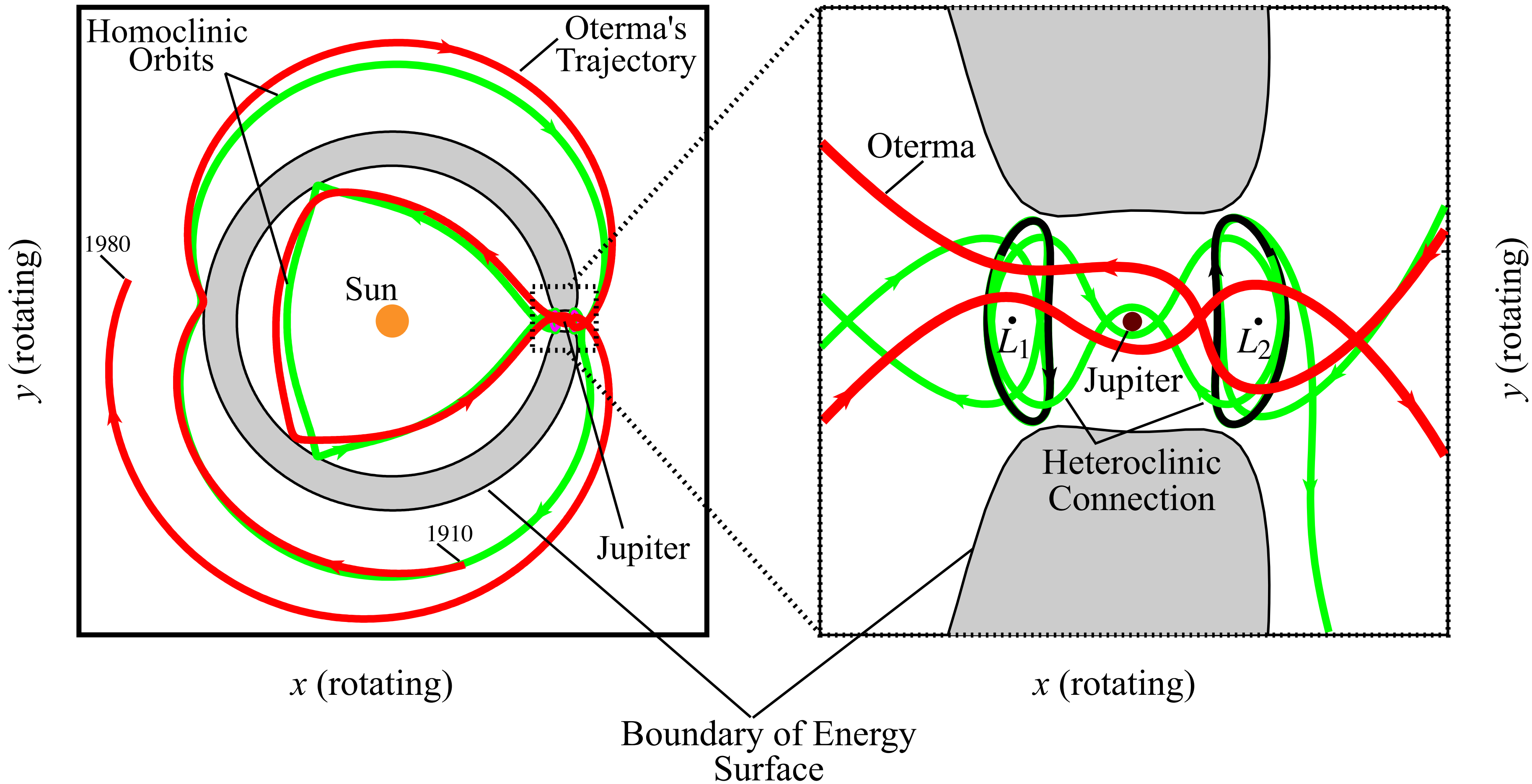
Jupiter Family Comets

- Rapid transition: outside to inside Jupiter's orbit.
  - Captured temporarily by Jupiter during transition.
  - Exterior (2:3 resonance) to interior (3:2 resonance).



Viewed in Rotating Frame

- **Oterma**'s orbit in rotating frame with some invariant manifolds of the 3-body problem superimposed.

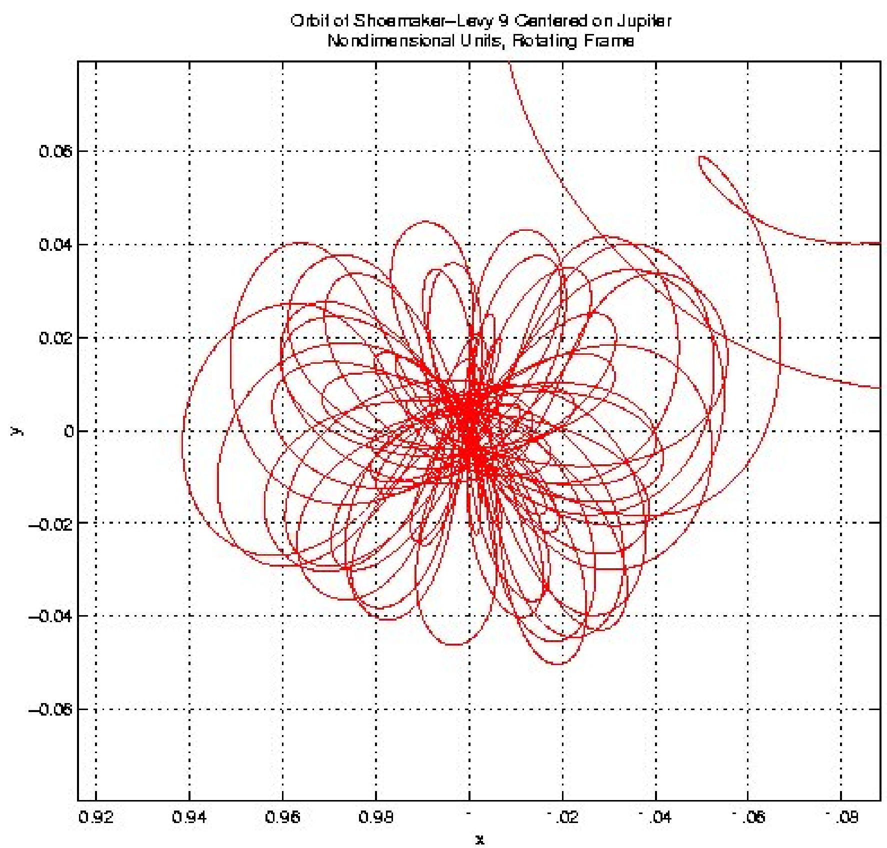


Viewed in Rotating Frame

Oterma - Rotating Frame

Collisions with Jupiter

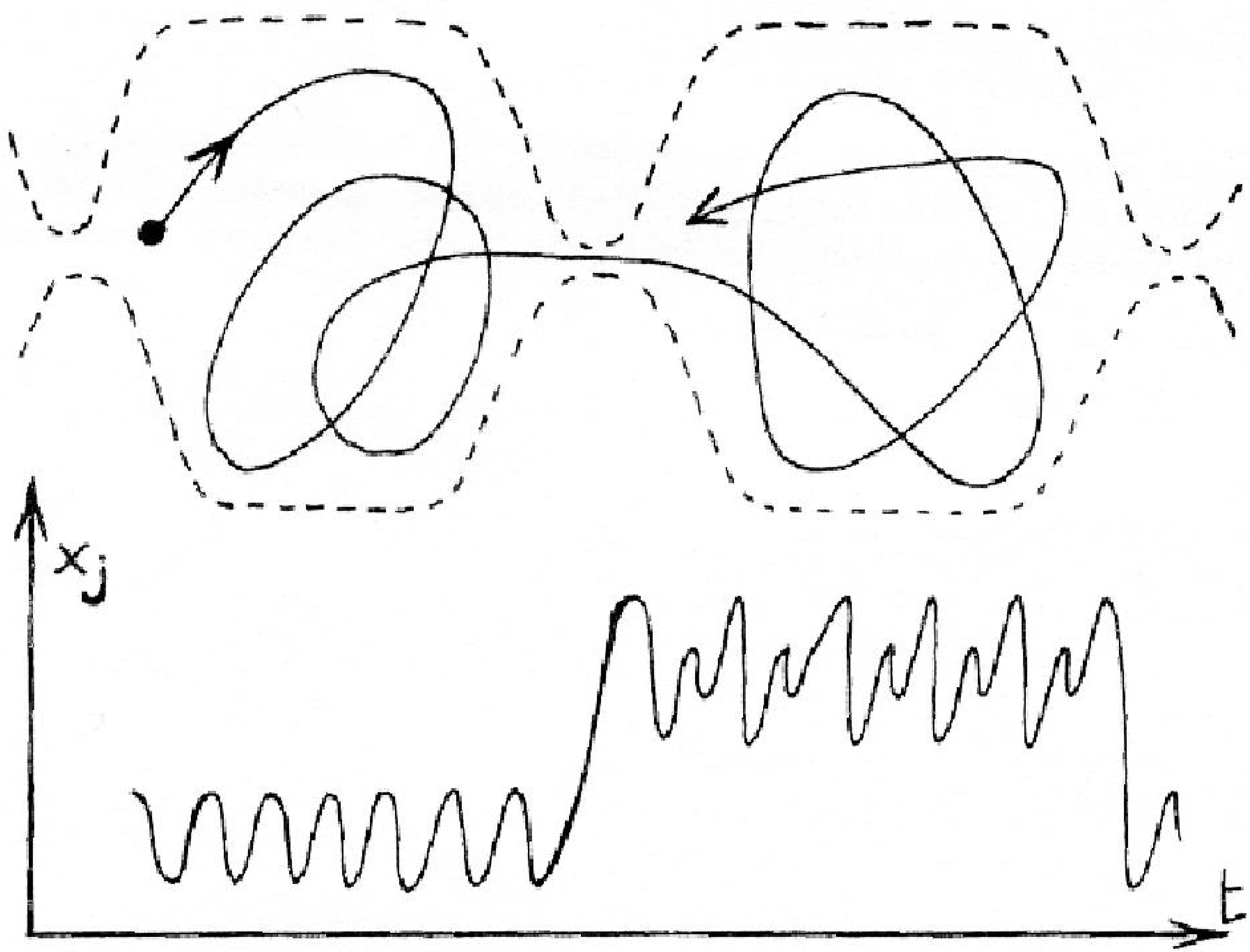
- **Shoemaker Levy-9**: similar energy to **Oterma**
  - Temporary capture and collision; came through L1 or L2



Possible *Shoemaker-Levy 9* orbit seen in rotating frame (Chodas, 2000)

Chaotic Dynamics

Transport through a **bottleneck** in phase space; intermittency



Transport Theory

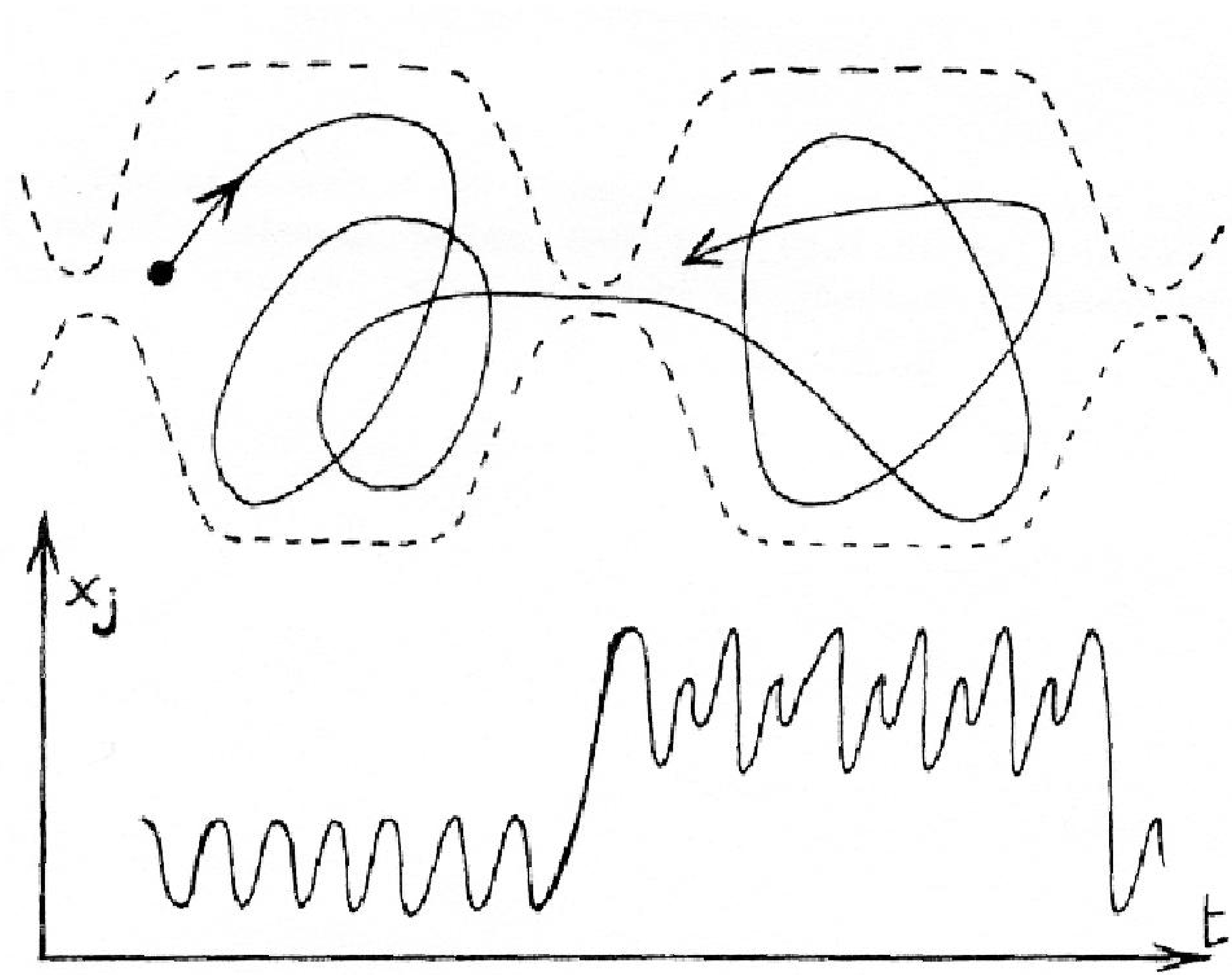
- *Chaotic dynamics*
  $\Longrightarrow$ ***statistical methods***
- ***Transport theory***
  - **Ensembles** of phase space trajectories
    - How long or likely to move from one region to another?
    - Determine transition probabilities
  - Applications:
    - Comet and asteroid collision probabilities, resonance transition probabilities, transport rates

Transport Theory

- ***Transport in the solar system***
  - For objects of interest
    - e.g., Jupiter family comets, near-Earth asteroids, dust
  - **Identify phase space objects** governing transport
  - View N-body as multiple restricted 3-body problems
  - Consider stable/unstable manifolds of bounded orbits associated with **libration points**
    - e.g, planar Lyapunov orbits
  - Use these to **compute statistical quantities**
    - e.g., probability of resonance transition, escape rates

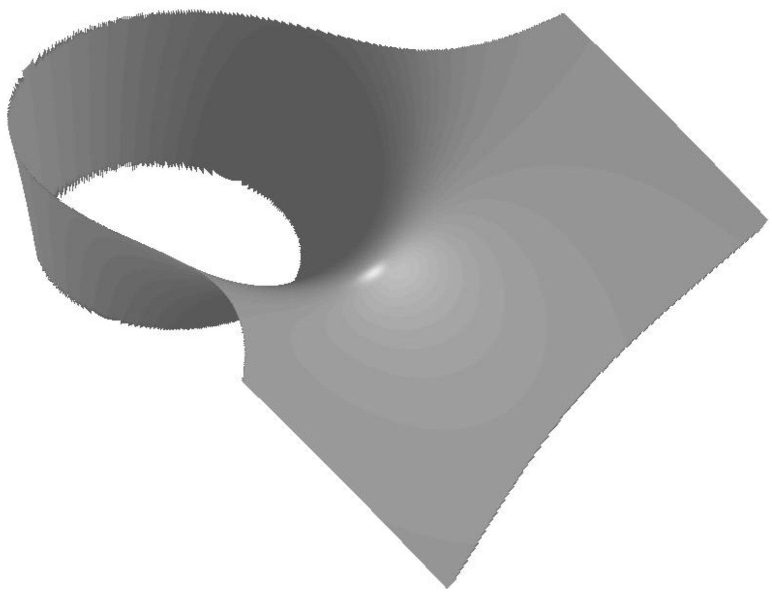
Partition the Phase Space

Region A **Region B**

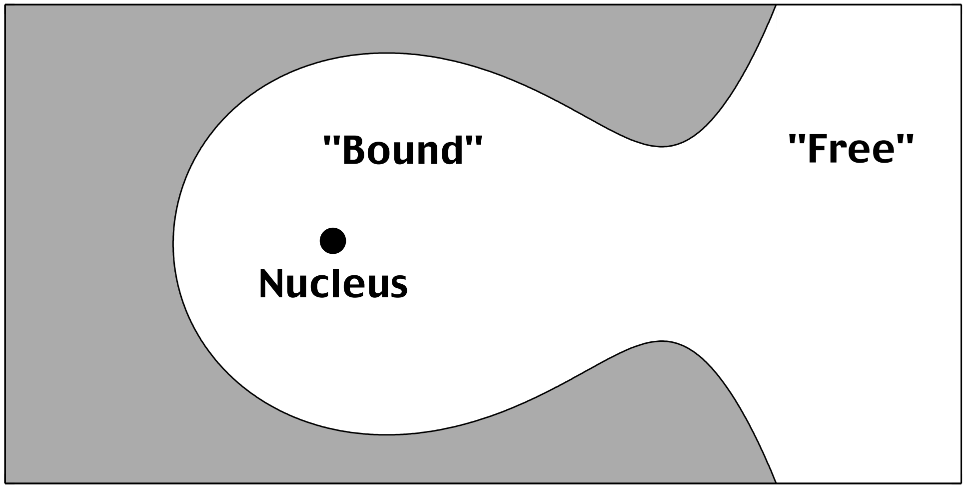


Partition the Phase Space

- ***Systems with potential barriers***
  - Electron near a nucleus with crossed electric and magnetic fields
    - See Jaffé, Farrelly, and Uzer [1999]



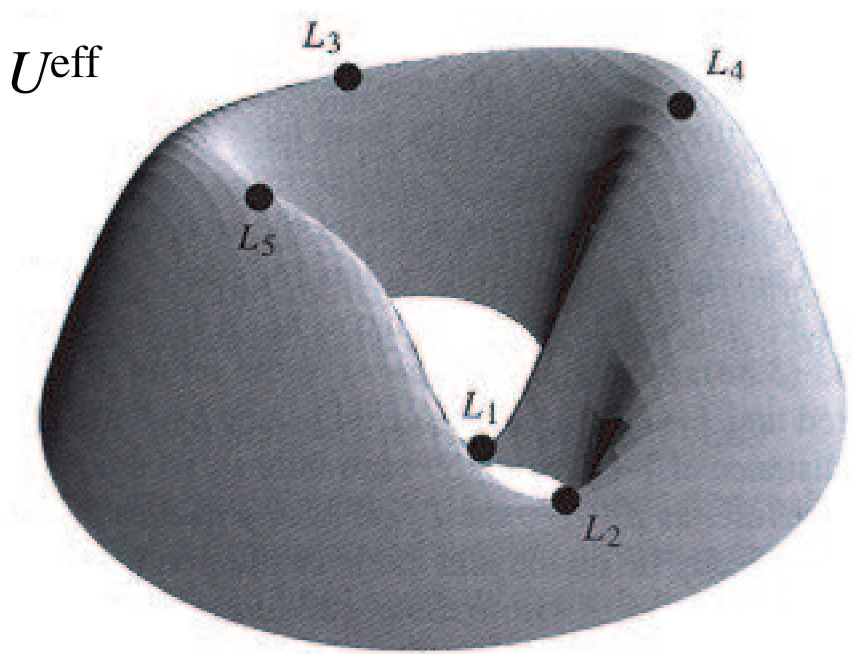
Potential



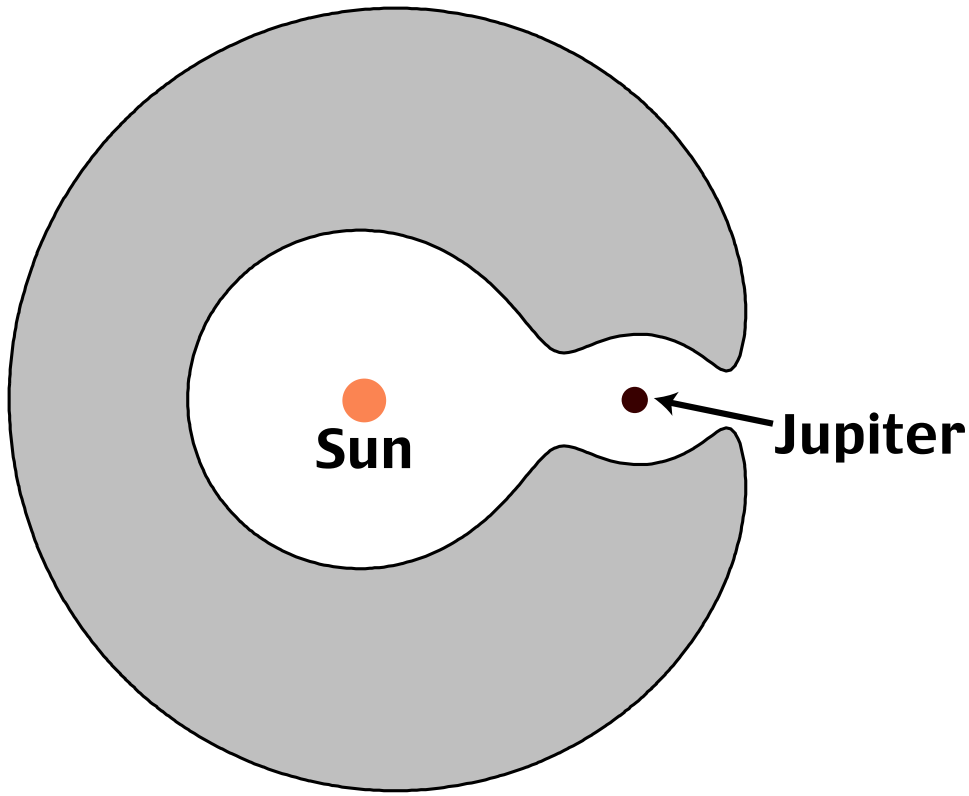
Configuration Space

Partition the Phase Space

- Comet near the Sun and Jupiter
  - Some behavior similar to electron!



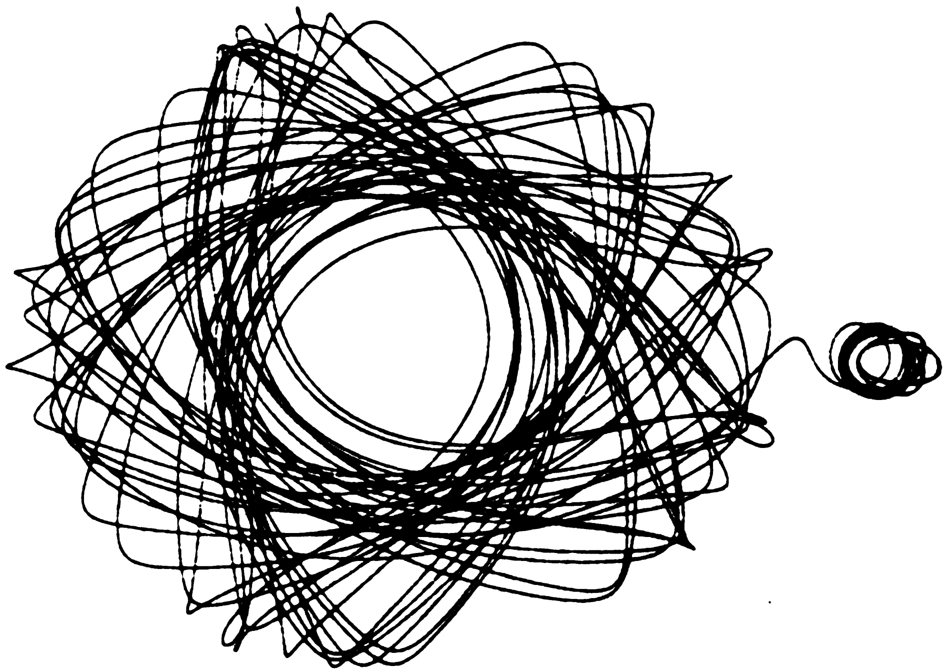
Potential



Configuration Space

Partition the Phase Space

- ***Partition is specific to problem***
  - We desire a way of describing dynamical boundaries that represent the **"frontier"** between qualitatively different types of behavior
- ***Example: motion of a comet***



- motion around the Sun
- motion around Jupiter

Statement of Problem

- Following Wiggins [1992]:
- Suppose we study the motion on a manifold $\mathcal{M}$
- Suppose $\mathcal{M}$ is partitioned into disjoint regions

  $$R_i, i = 1, \ldots, N_R,$$

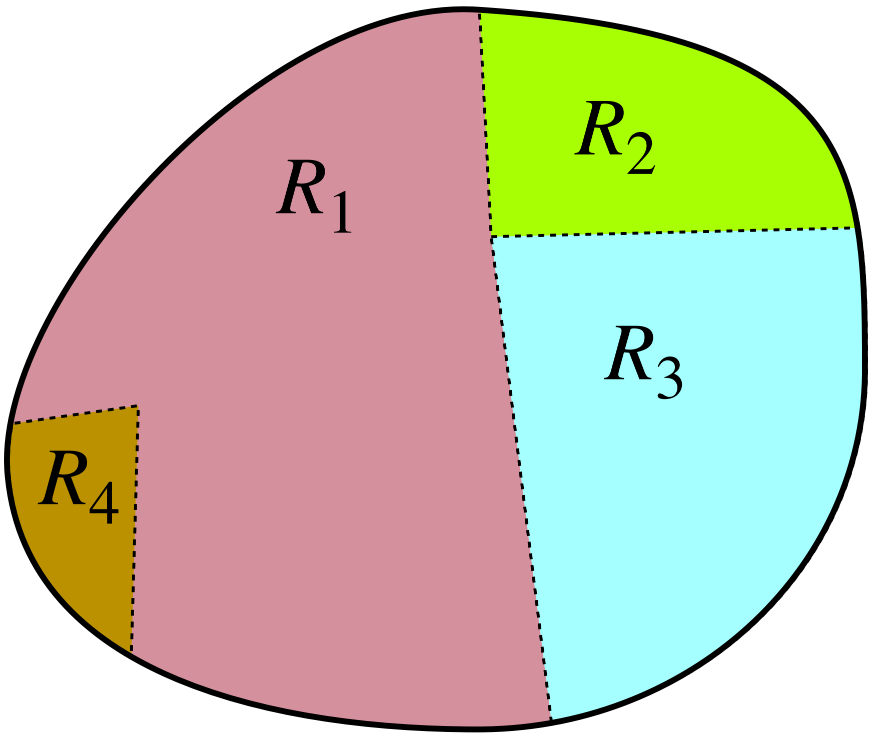
  such that

  $$\mathcal{M} = \bigcup_{i=1}^{N_R} R_i.$$

- At $t = 0$, **region** R_i is **uniformly covered** with **species** S_i
- Thus, species type of a point indicates the region in which it was **located initially**

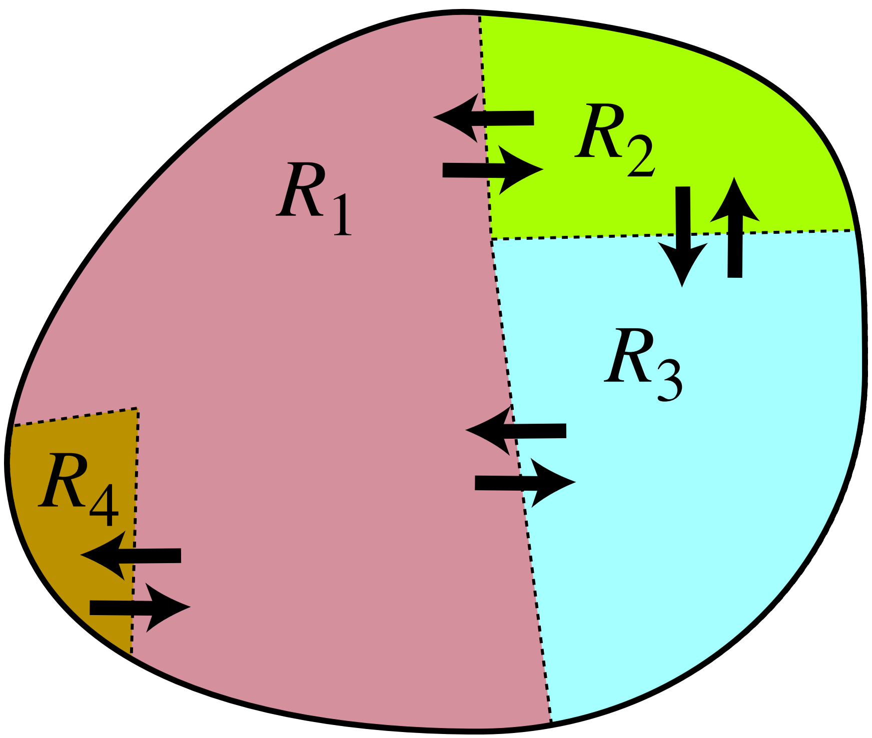
Statement of Problem

- Statement of the transport problem:
  Describe the distribution of species $S_i, i = 1, \ldots, N_R$, throughout the regions $R_j, j = 1, \ldots, N_R$, for any time $t > 0$.



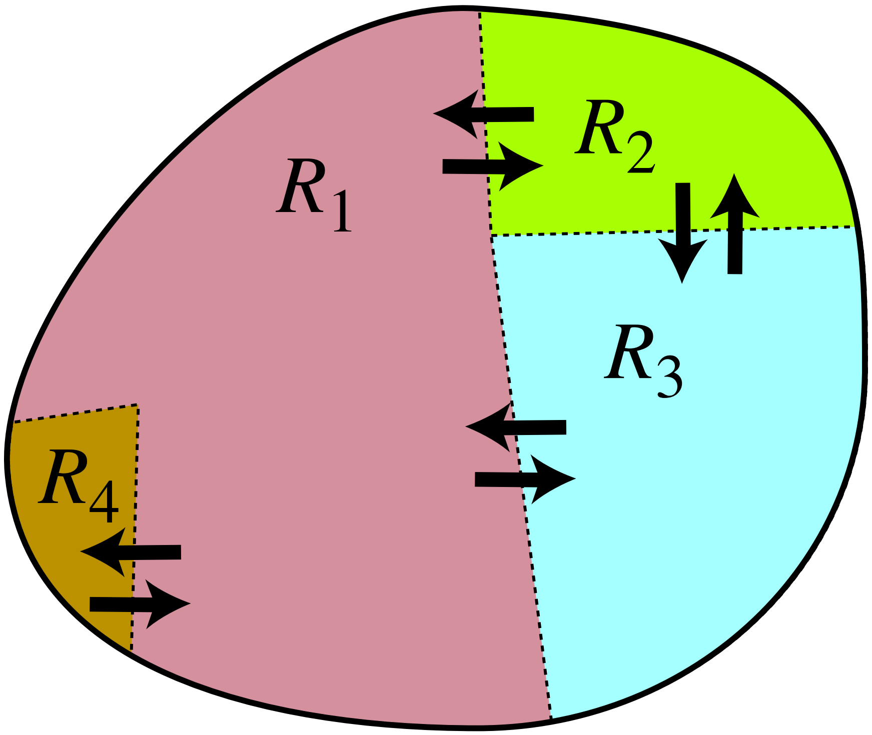
Statement of Problem

- Some quantities we would like to compute are:
  $T_{i,j}(t) =$ amount of species S_i contained in region R_j
  $F_{i,j}(t) = \frac{dT_{i,j}}{dt}(t) =$ flux of species S_i into region R_j



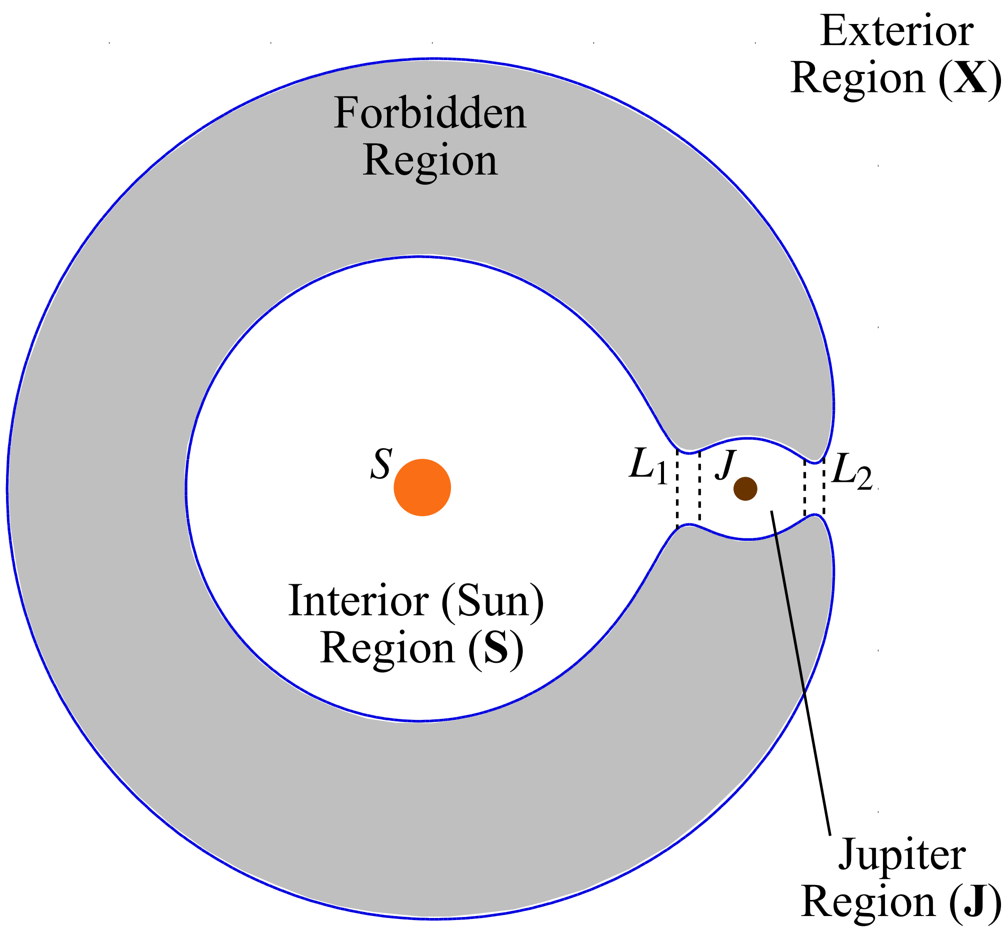
Probabilities

- For some problems, probability more relevant
  - e.g., probability $= 0$ implies event should not occur
- Test this on celestial mechanics problems of interest



Restricted 3-Body Prob.

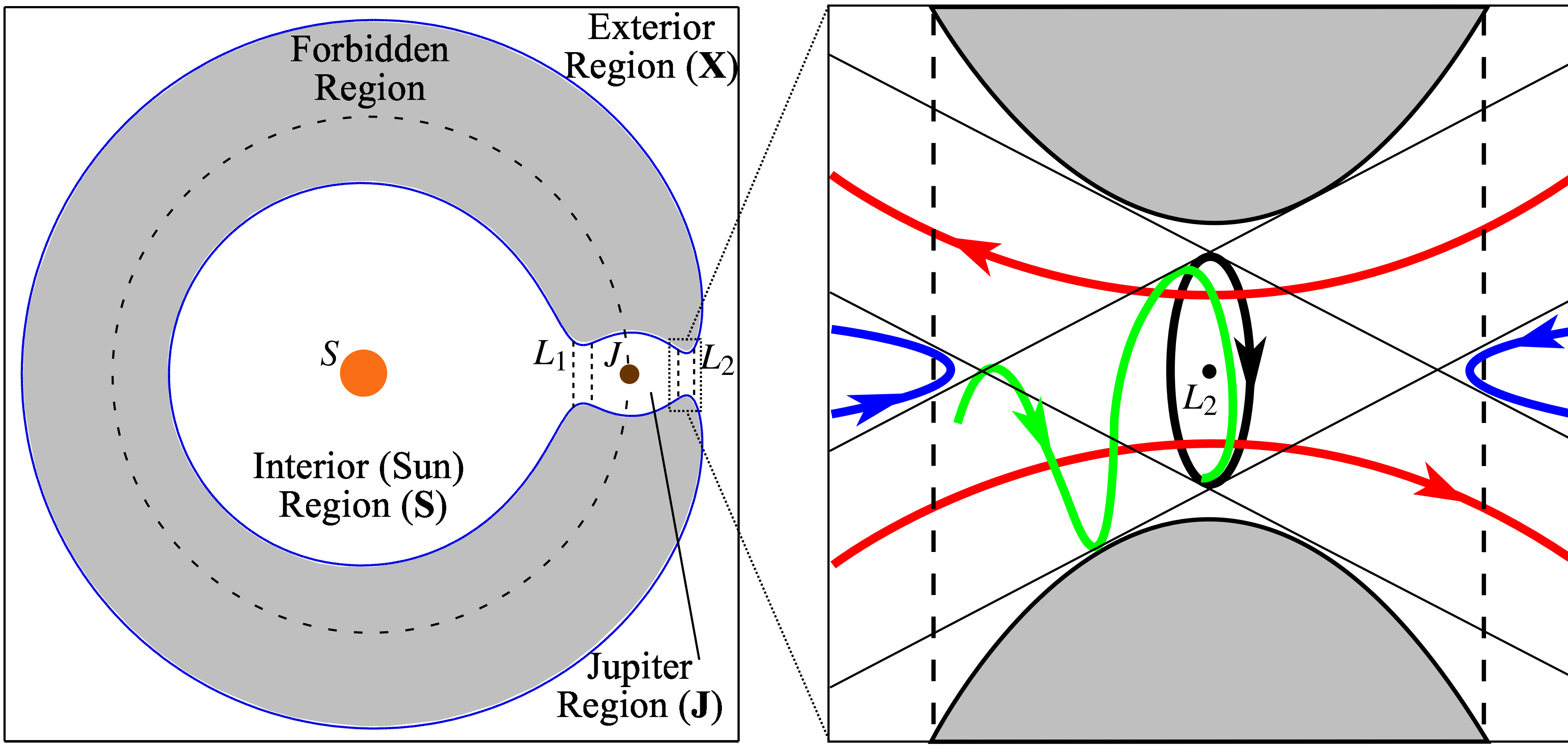
- ***Planar circular case***
  - Partition the energy surface: **S, J, X regions**



Position Space Projection

Equilibrium Region

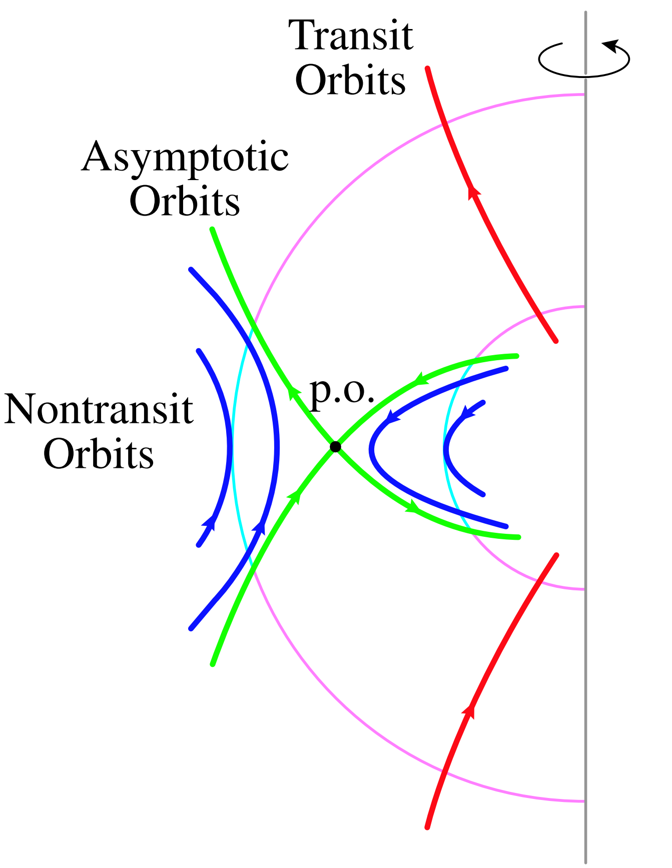
- ***Look at motion near the potential barrier, i.e. the equilibrium region***



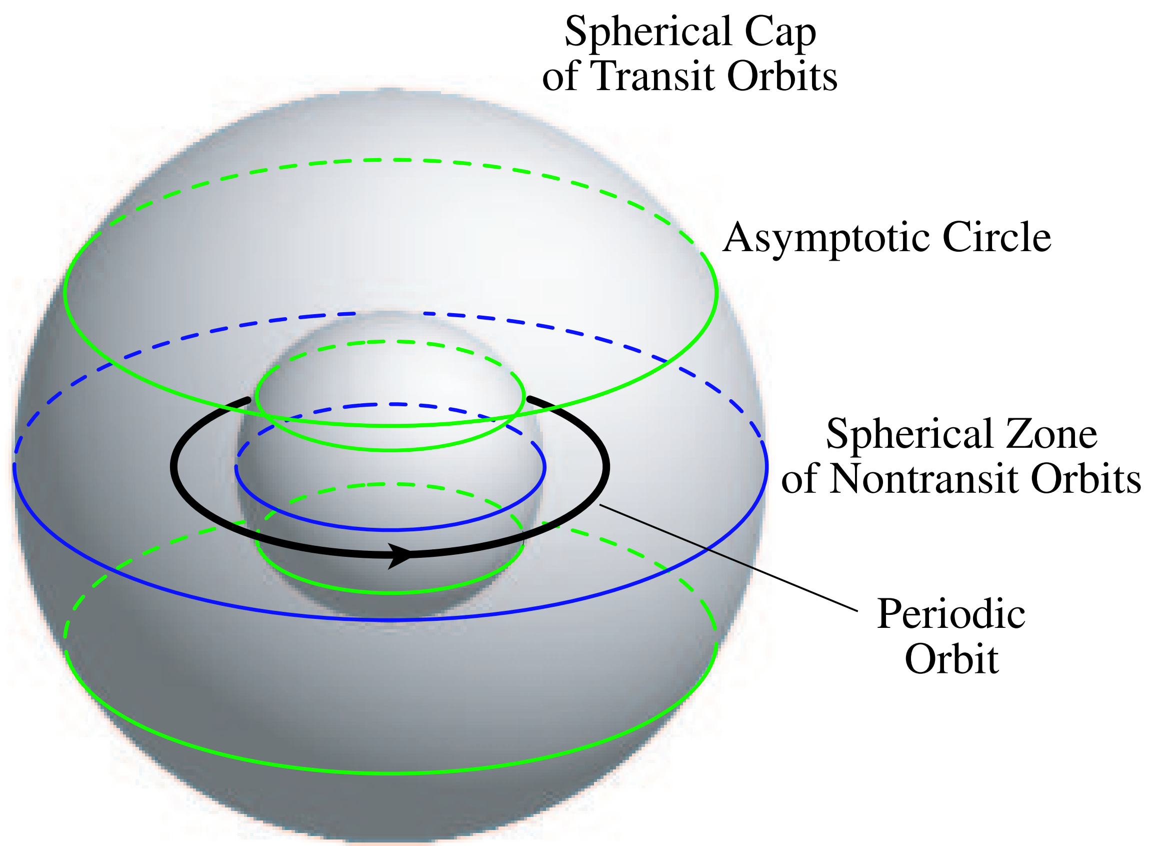
Position Space Projection

Local Dynamics

- For fixed energy, the equilibrium region $\simeq S^2 \times \mathbb{R}$.
  - Stable/unstable manifolds of periodic orbit define mappings between bounding spheres on either side of the barrier



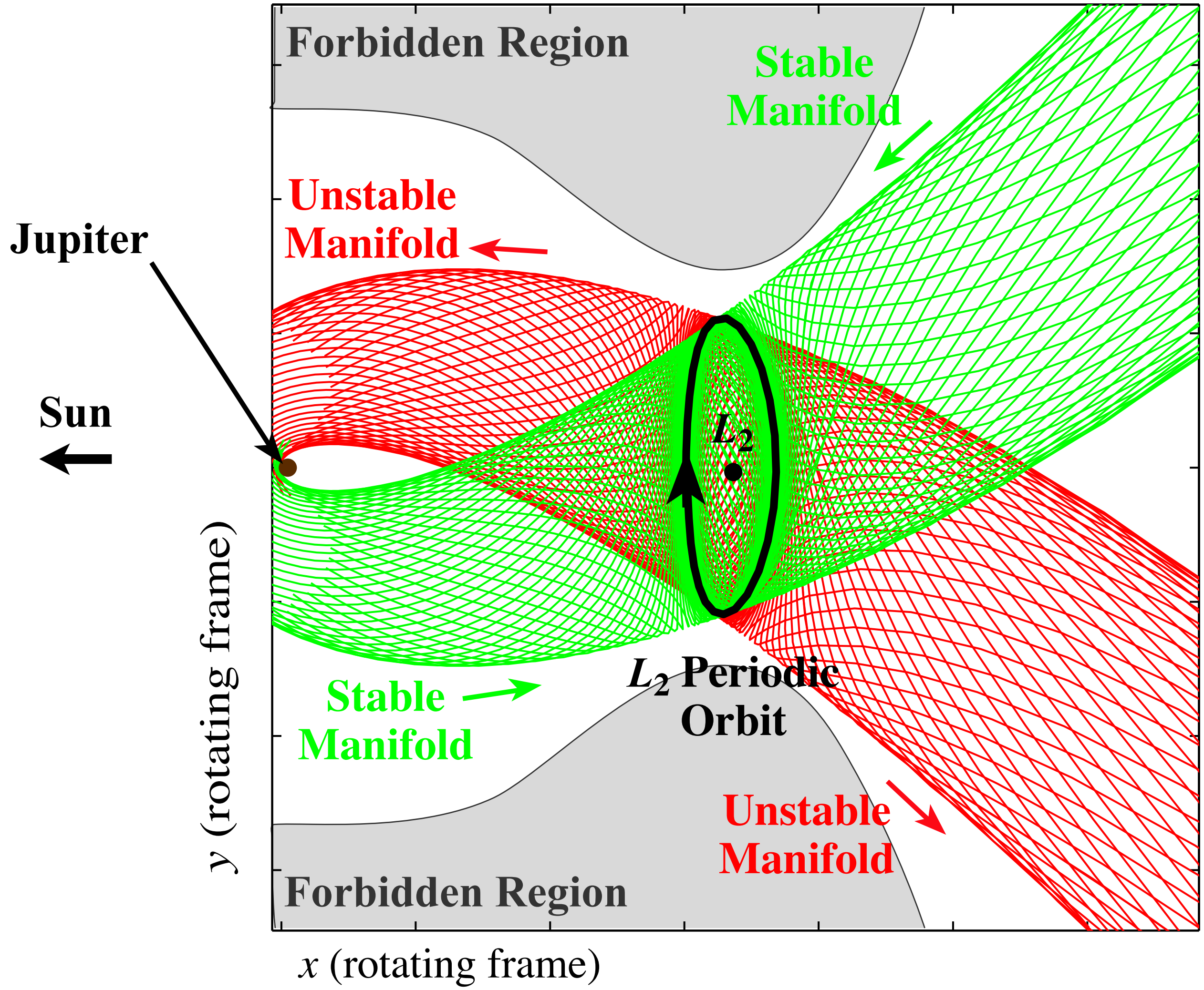
Cross-section of Equilibrium Region



Equilibrium Region

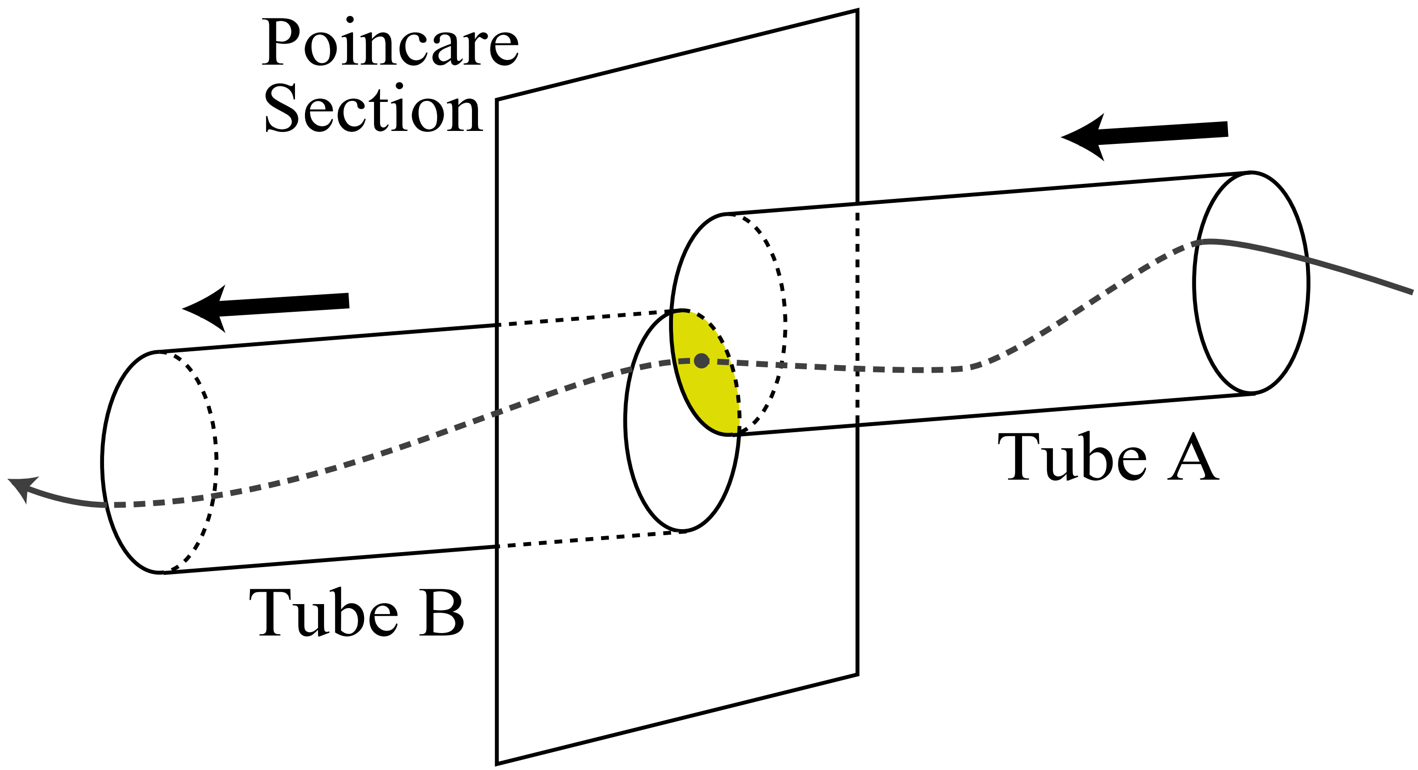
Tubes in the 3-Body Problem

- Stable and unstable manifold tubes
  - Control transport through the potential barrier.



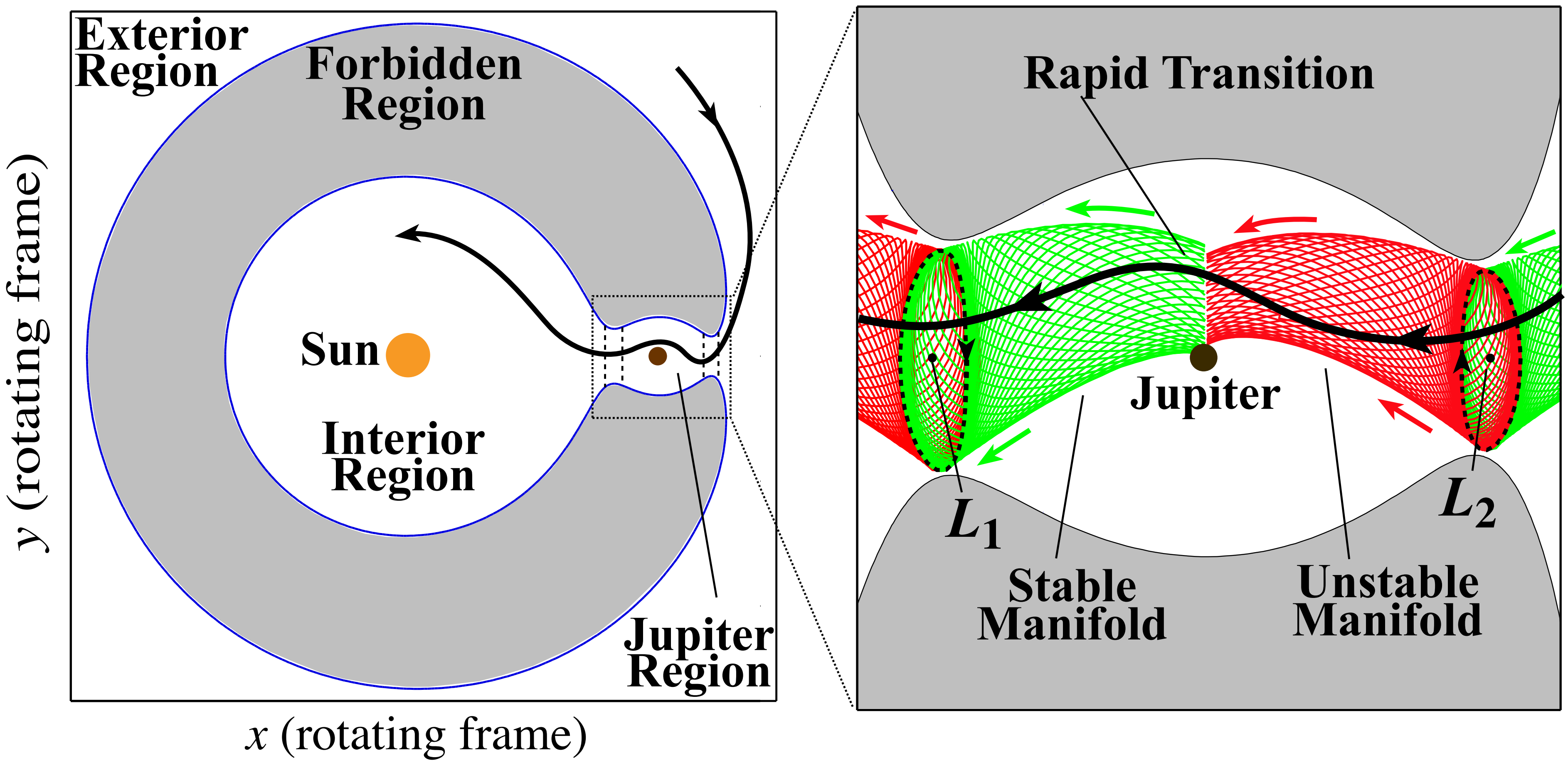
Transition Probabilities

- ***Transport btwn non-adjacent regions***
  - Consider **intersections** between the interior of tubes — the transit orbits connecting regions.
  - Tube A and Tube B from different potential barriers.



Transition Probabilities

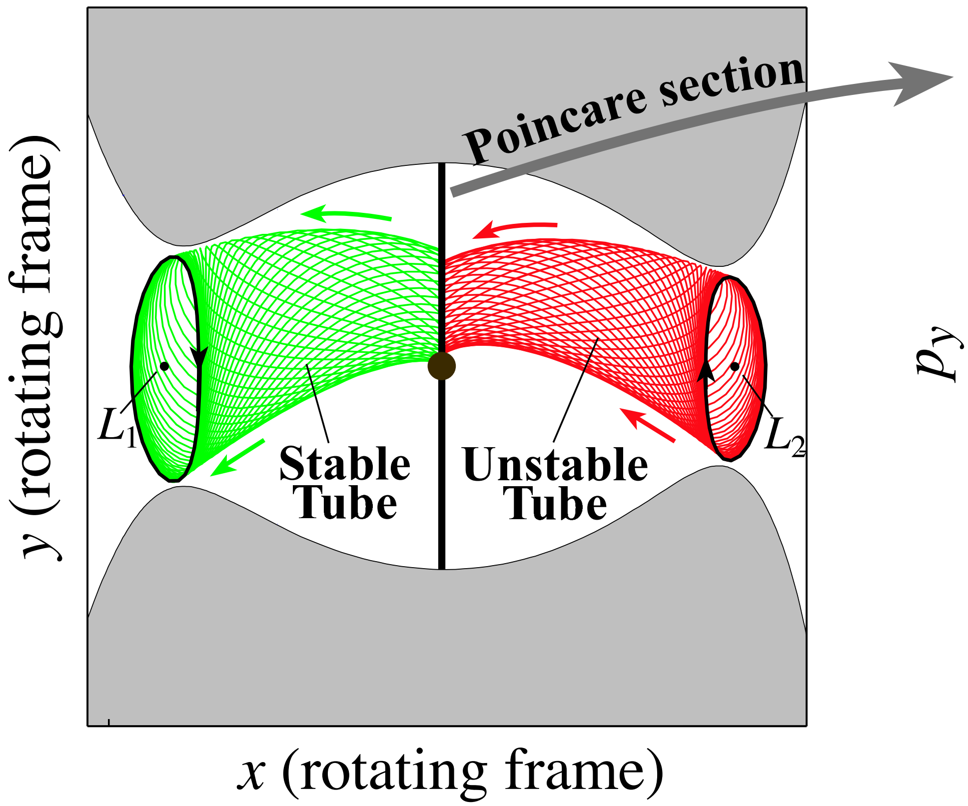
- Example: Comet transport between outside and inside of Jupiter (i.e., **Oterma**-like transitions)



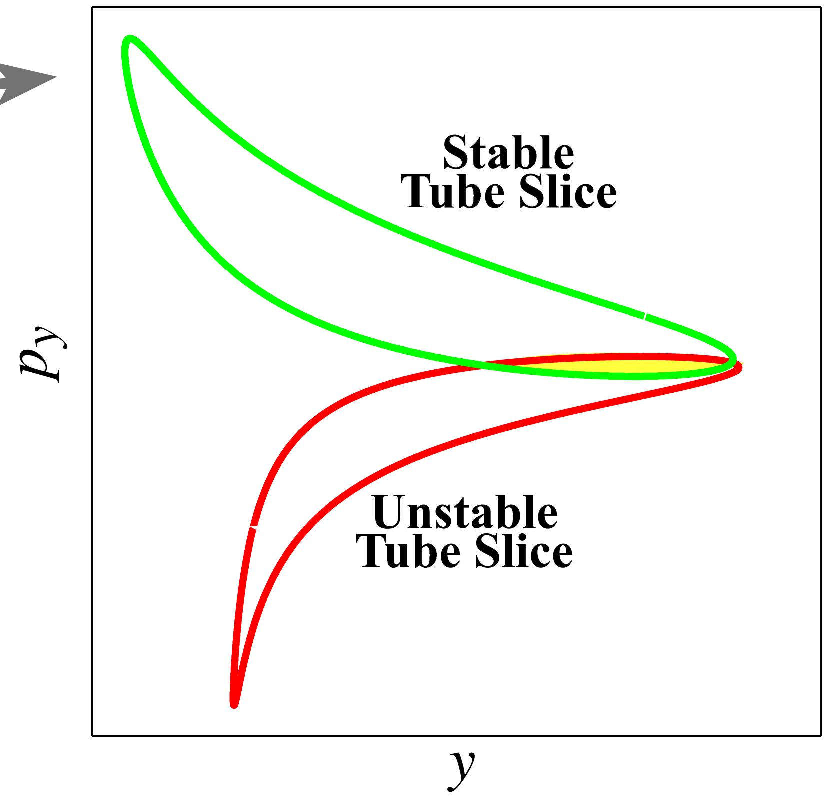
(a)

Transition Probabilities

- Consider Poincaré section intersected by both tubes.
- Choosing surface $\{x = \text{constant}; p_x < 0\}$, we look at the canonical plane (y, p_y).



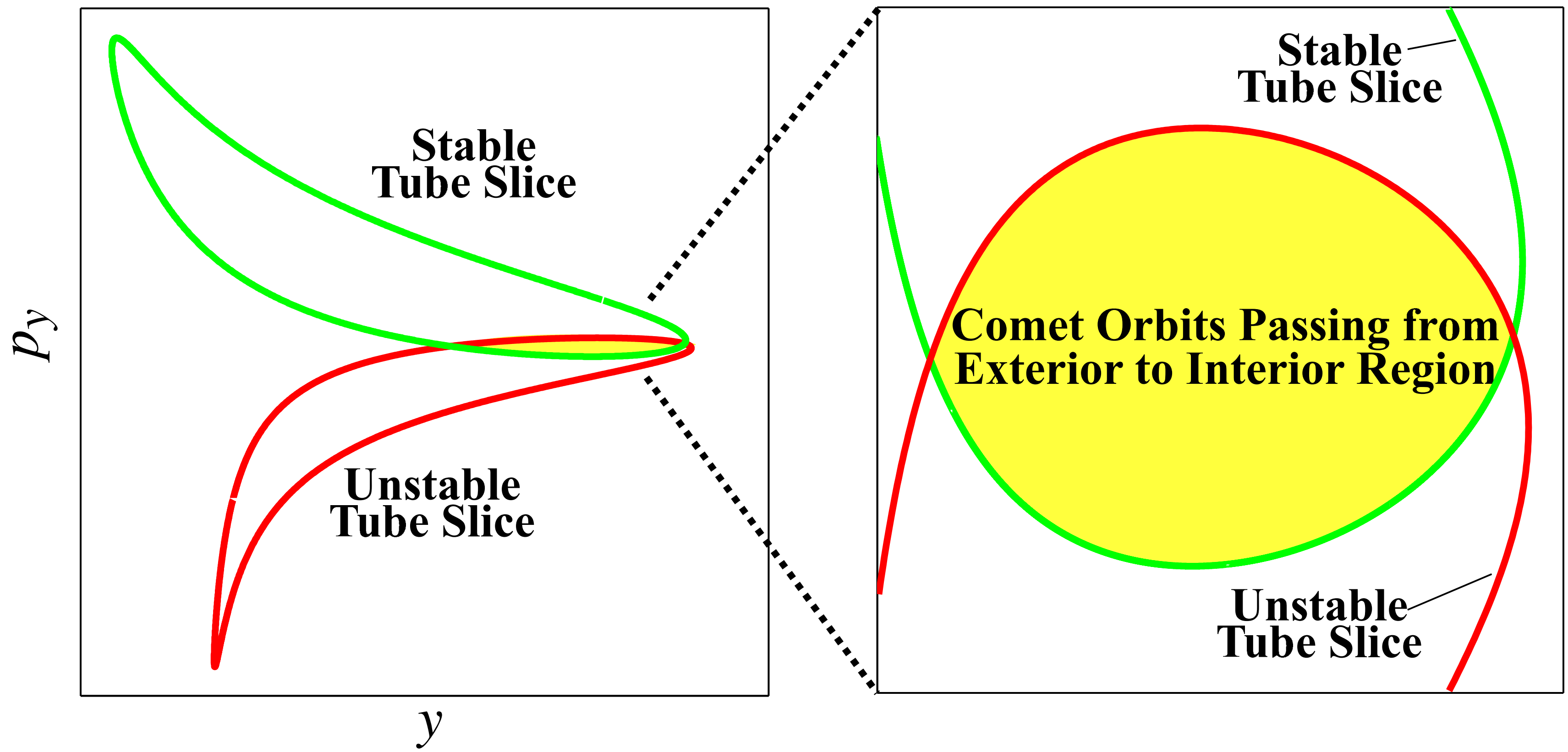
Position Space



Canonical Plane (y, p_y)

Transition Probabilities

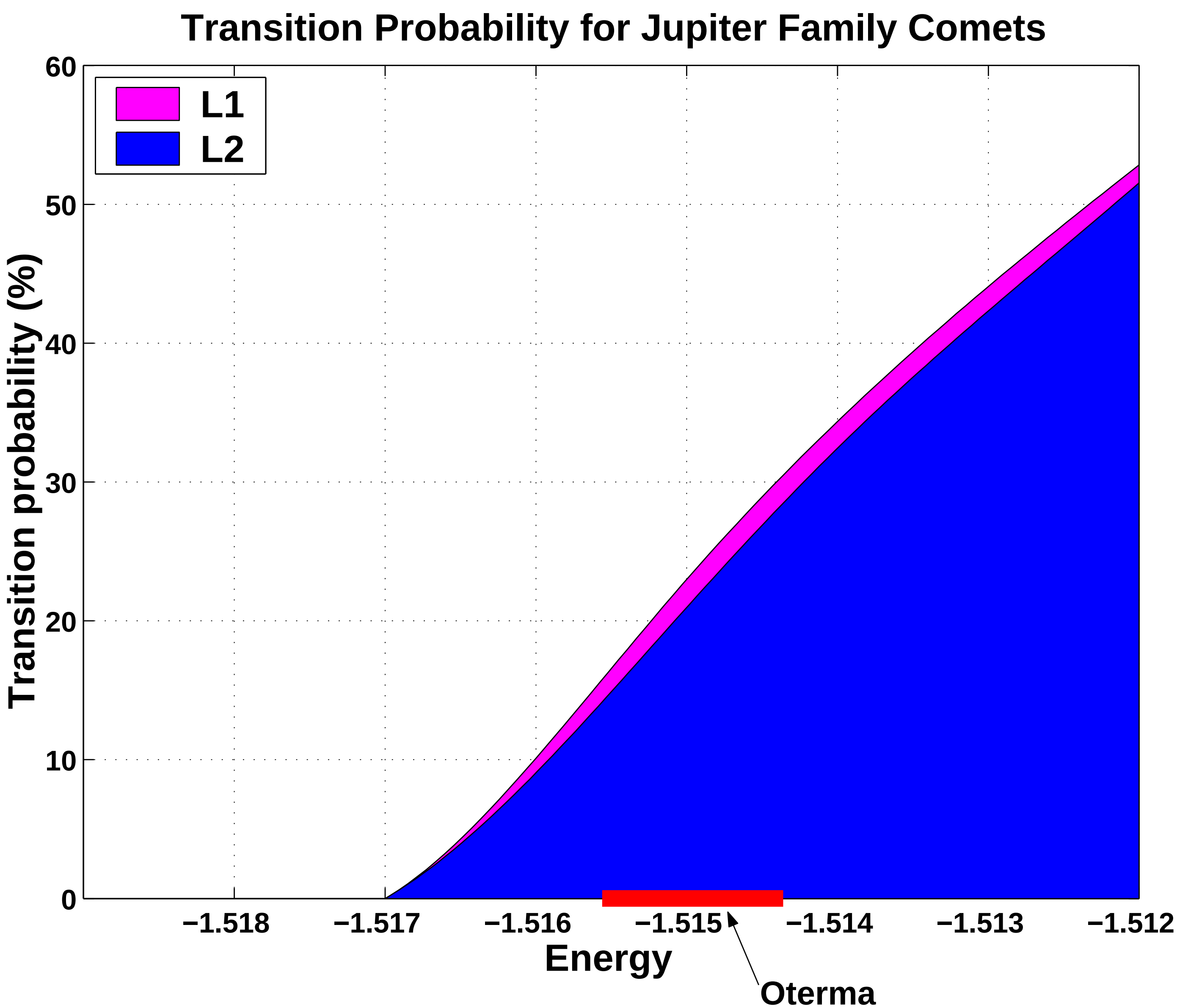
- Canonical area ratio gives the **conditional probability** to pass from **outside** to **inside** Jupiter's orbit.
  - Assuming a well-mixed connected region on the energy mfd.



Canonical Plane (y, p_y)

Transition Probabilities

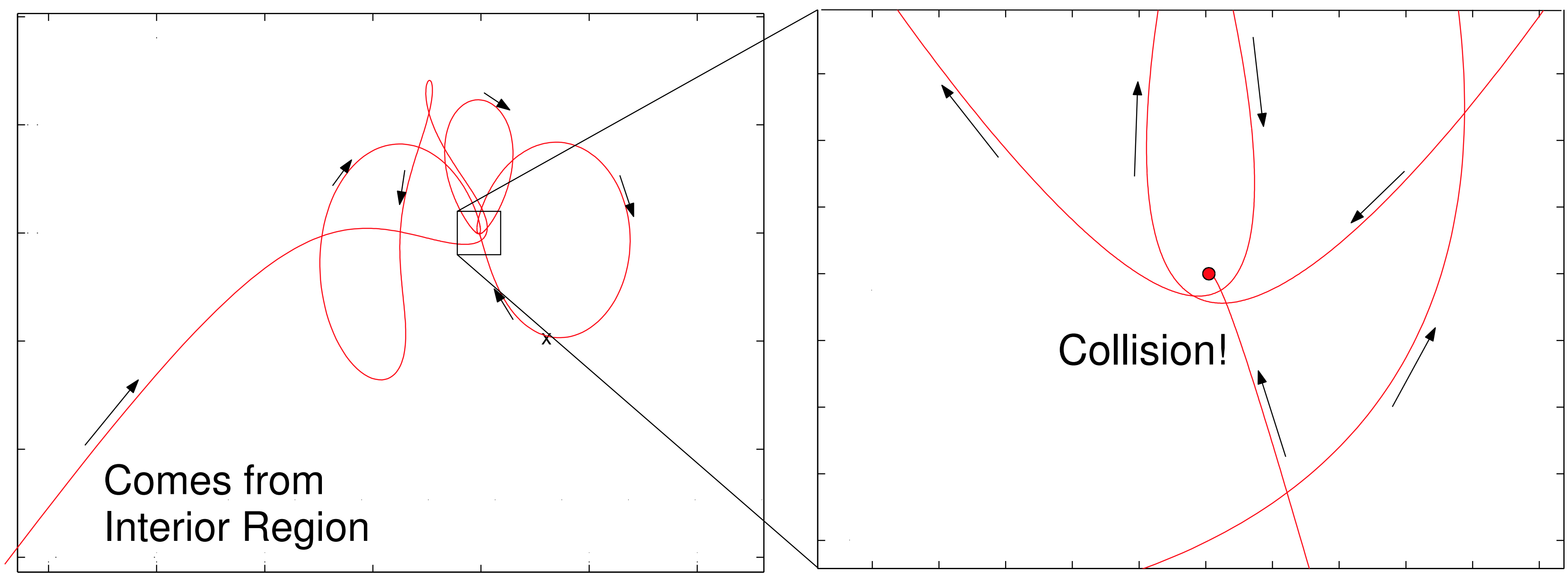
- Jupiter family comet transitions: **X** → **S**, **S** → **X**



Collision Probabilities

- Low velocity impact probabilities
- Assume object enters the planetary region with an energy slightly above L1 or L2
  - eg, **Shoemaker-Levy 9** and **Earth-impacting asteroids**

Example Collision Trajectory



Collision Probabilities

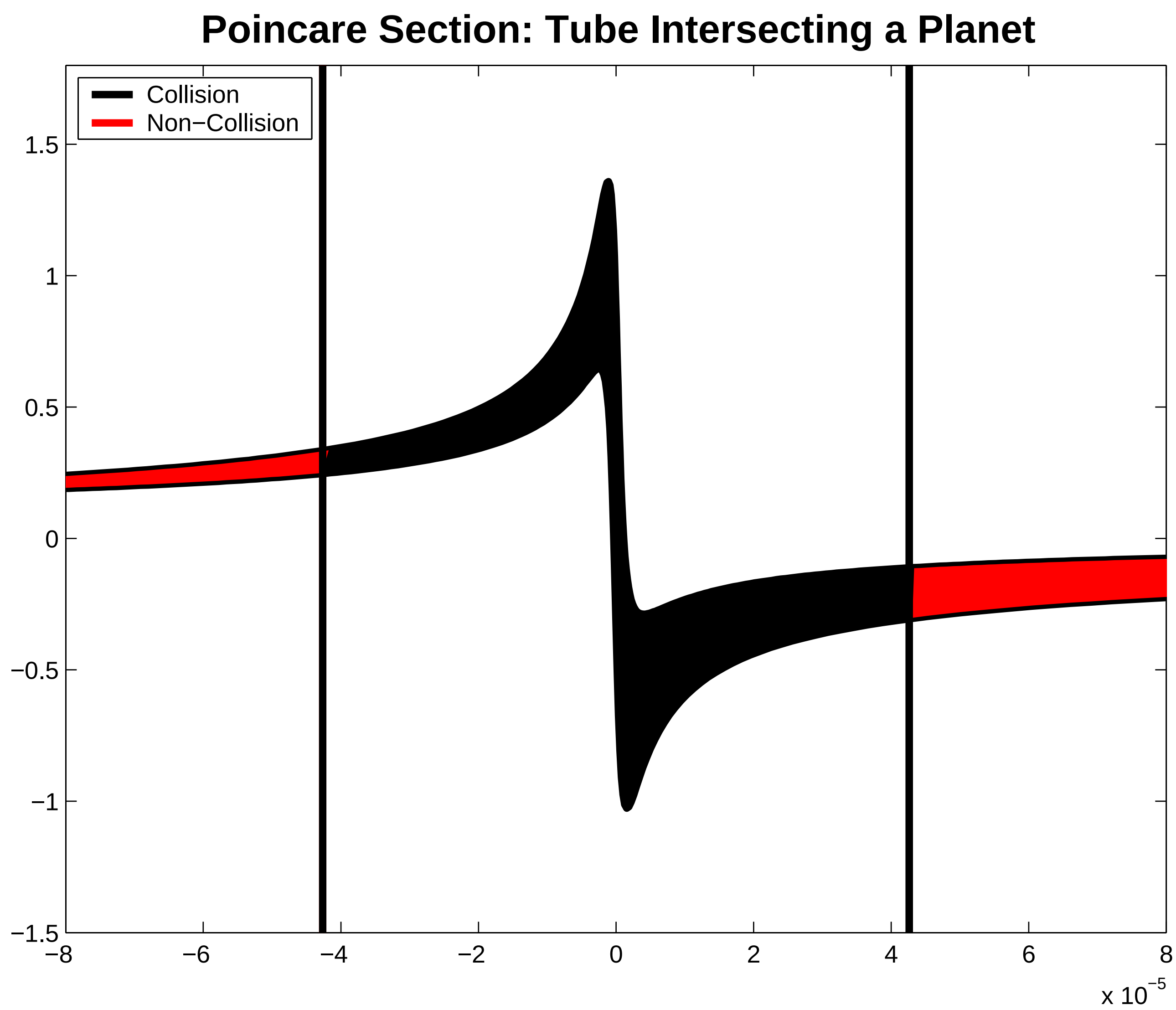
Collision probabilities

- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter, in addition to μ and energy E

$\longleftarrow$ Diameter of planet $\longrightarrow$

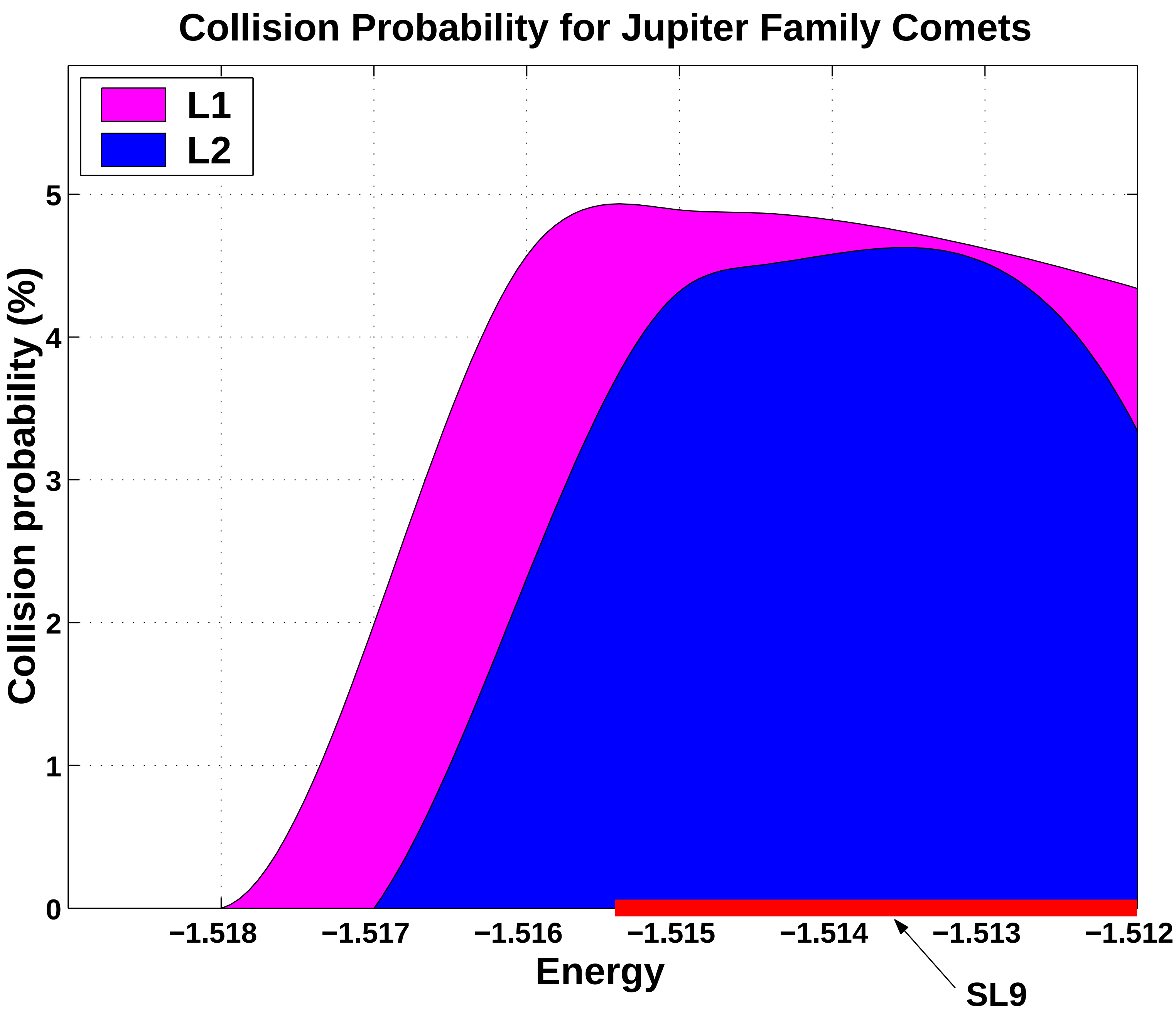
Collision Probabilities

- *Collision probabilities*

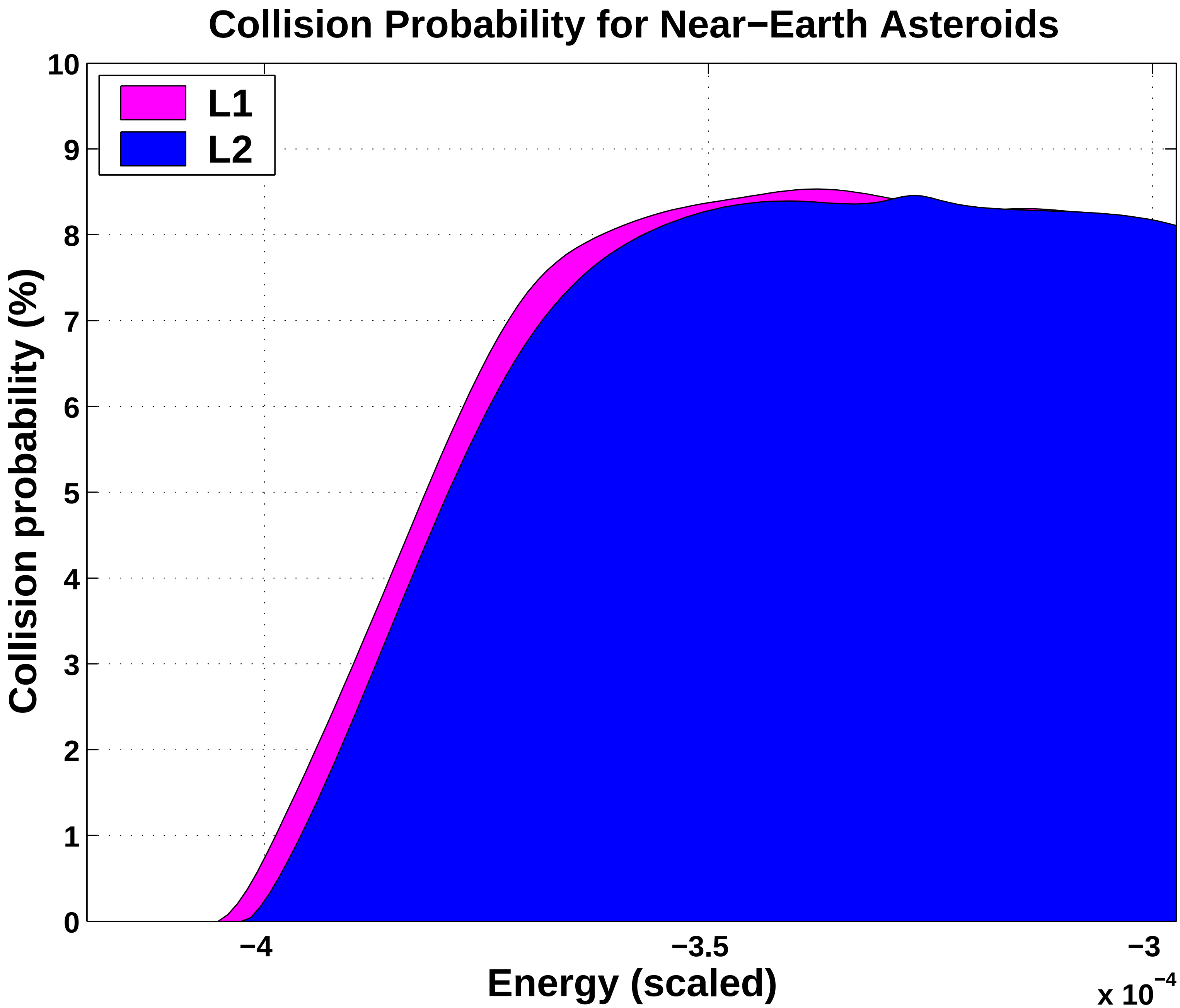


← Diameter of planet →

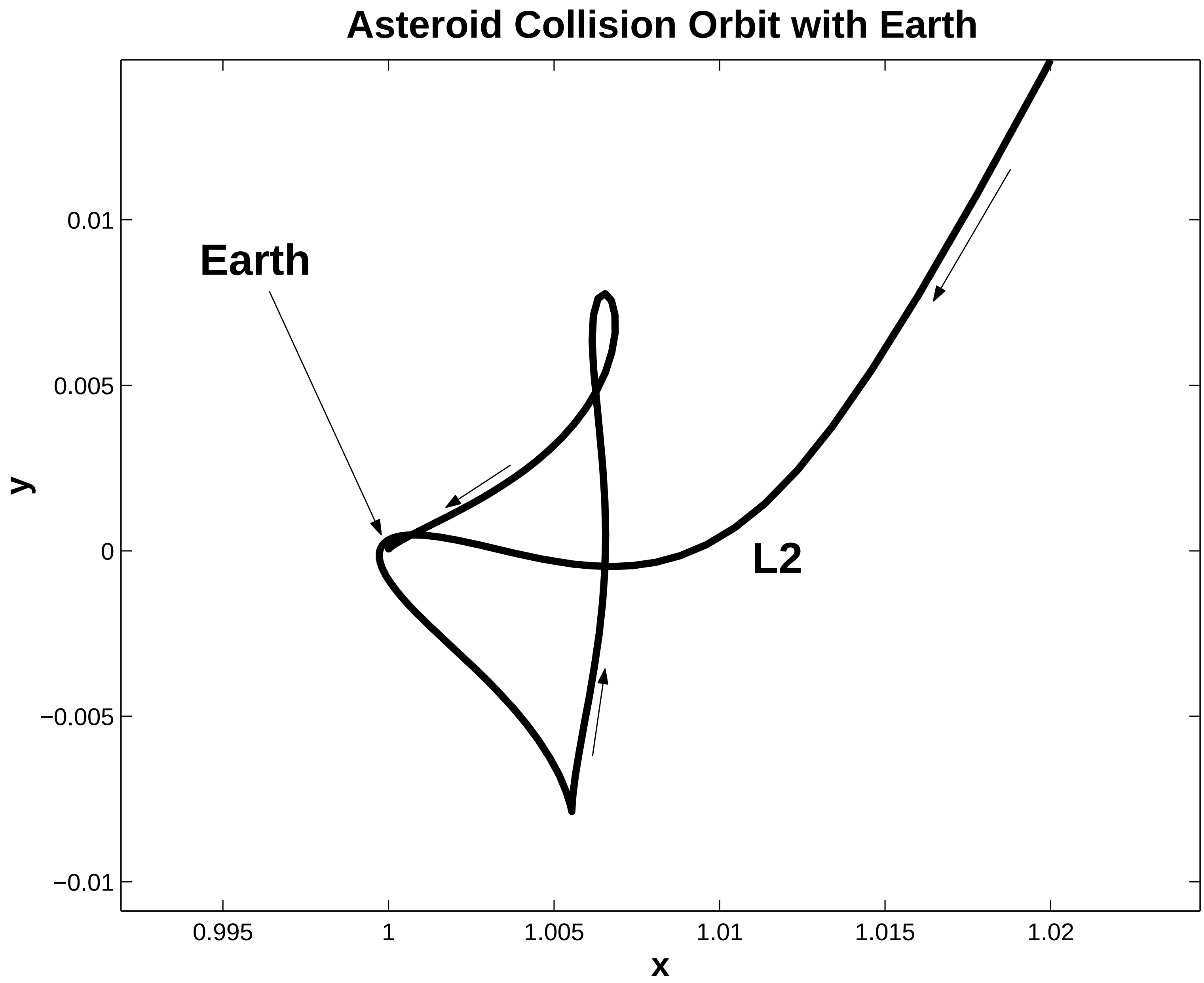
Collision Probabilities



Collision Probabilities



Collision Probabilities



Conclusion and Future Work

- ***Transport in the solar system***
  - Approximate some solar system phenomena using the restricted 3-body problem
  - Planar restricted 3-body problem
    - Stable and unstable manifold tubes of libration point orbits can be used to compute statistical quantities of interest
    - Probabilities of transition, collision
  - Theory and observation agree
- ***Future studies to involve multiple three-body problems***

References

- Jaffé, C., S.D. Ross, M.W. Lo, J. Marsden, D. Farrelly, and T. Uzer [2002] ***Statistical theory of asteroid escape rates***. *Physical Review Letters*, to appear.
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- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] ***Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics.*** *Chaos* 10(2), 427–469.

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The End